The equivalence between many-to-one polygraphs and opetopic sets

Cédric Ho Thanh[1]

July 7th, 2018

[1]IRIF, Paris Diderot University, INSPIRE 2017 Fellow, This project has received funding from the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement No 665850

This short talk informally presents the main notions and results of [HT, 2018] (`arXiv:1806.08645 [math.CT]`).

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1. Polygraphs
2. Opetopes
3. Main result and ideas of how to prove it
4. Conclusion

Polygraphs

Idea

Given a graph

$$G = \left(G_0 \xleftarrow{s,t} G_1 \right),$$

one can generate the *free category* G^*:

Objects vertices of G;

Generating morphisms edges of G;

Relations none.

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one can generate the *free category* G^*:

Objects vertices of G;

Generating morphisms edges of G;

Relations none.

In the same way, an *n-polygraph* (also called *n-computad*) generates a free (strict) n-category, for $n \leq \omega$.

Definition

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A 1-*polygraph* P is a graph

$$P = \left(P_0 \xleftarrow{s,t} P_1 \right).$$

It generates a 1-category P^* which is the free category on P.

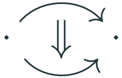
Definition

An $(n+1)$-*polygraph* P is the data of an n-polygraph Q, a set P_{n+1}, and two maps

$$Q_n^* \xleftarrow{\mathrm{s,t}} P_{n+1}$$

such that the globular identities hold: for $p \in P_{n+1}$

$$\mathrm{s}\,\mathrm{s}\,p = \mathrm{s}\,\mathrm{t}\,p, \qquad \mathrm{t}\,\mathrm{s}\,p = \mathrm{t}\,\mathrm{t}\,p.$$



Definition

The $(n+1)$-category P^* is defined as follows:

1. its underlying n-category is Q^* (i.e. the n-category generated by the underlying n-polygraph Q of P), so that $P^*_k = Q^*_k$ for $k \leq n$;
2. its $(n+1)$-cells are the formal composites of elements of P_{n+1} according to $Q^*_n \xleftarrow{s,t} P_{n+1}$, as well as identities of cells of Q^*.

Definition

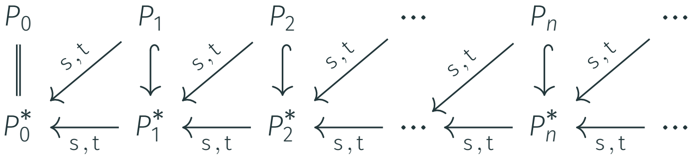
Thus the $(n+1)$-polygraph P can be depicted as follows:

$$
\begin{array}{ccccccccccc}
P_0 & & P_1 & & P_2 & & \cdots & & P_n & & P_{n+1} \\
\| & \swarrow^{\mathrm{s},\mathrm{t}} & \downarrow & \swarrow^{\mathrm{s},\mathrm{t}} & \downarrow & \swarrow^{\mathrm{s},\mathrm{t}} & & \swarrow^{\mathrm{s},\mathrm{t}} & \downarrow & \swarrow^{\mathrm{s},\mathrm{t}} & \downarrow \\
P_0^* & \underset{\mathrm{s},\mathrm{t}}{\longleftarrow} & P_1^* & \underset{\mathrm{s},\mathrm{t}}{\longleftarrow} & P_2^* & \underset{\mathrm{s},\mathrm{t}}{\longleftarrow} & \cdots & \underset{\mathrm{s},\mathrm{t}}{\longleftarrow} & P_n^* & \underset{\mathrm{s},\mathrm{t}}{\longleftarrow} & P_{n+1}^*
\end{array}
$$

The maps s are called *source maps*, and t *target maps*. Elements of P_k are called *k-generators*, while elements of P_k^* are called *k-cells*. The bottom row is exactly the underlying globular set of P^*.

Definition

A *ω-polygraph* (or simply *polygraph*) P is a sequence $(P_{(n)} \mid n < \omega)$ such that $P_{(n)}$ is an n-polygraph that is the underlying n-polygraph of $P_{(n+1)}$.



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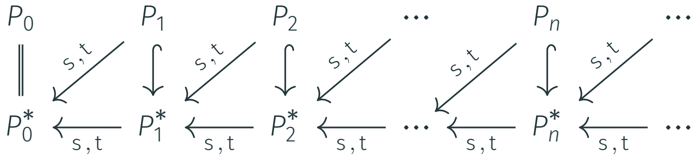

The underlying ω-category P^* is defined as

$$P^* = \operatorname{colim}\left(P^*_{(0)} \hookrightarrow P^*_{(1)} \hookrightarrow \cdots\right).$$

Definition

A *morphism of polygraphs* $f : P \longrightarrow R$ is an ω-functor $P^* \longrightarrow R^*$ mapping generators to generators. Let $\mathcal{P}\mathrm{ol}$ be the category of polygraphs and such morphisms, and $\mathcal{P}\mathrm{ol}_n$ be the full subcategory of $\mathcal{P}\mathrm{ol}$ spanned by n-polygraphs.

Proposition

The categories $\mathcal{P}\mathrm{ol}_0$, $\mathcal{P}\mathrm{ol}_1$, and $\mathcal{P}\mathrm{ol}_2$ are presheaf categories.

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Proposition [Cheng, 2013]

The category $\mathcal{P}\mathrm{ol}_3$ is not. Thus $\mathcal{P}\mathrm{ol}_n$ for $n \geq 3$, and $\mathcal{P}\mathrm{ol}$ aren't presheaf categories either.

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Which subcategories of $\mathcal{P}\mathrm{ol}$ are presheaf categories?

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Question

Which subcategories of $\mathcal{P}\mathrm{ol}$ are presheaf categories?

Answer (sort of)

A fair amount. See [Henry, 2017].

Many-to-one polygraphs

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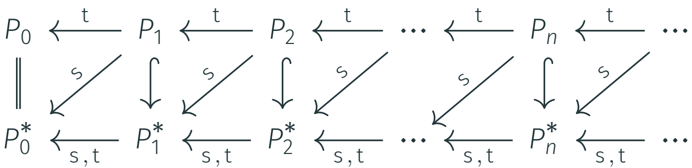

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Teaser

The category $\mathcal{P}\mathrm{ol}^{\triangledown}$ is a presheaf category.

Opetopes

Idea

Opetopes were originally introduced by Baez and Dolan in [Baez and Dolan, 1998] as an algebraic structure to describe compositions and coherence laws in weak higher dimensional categories.

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They have been reworked in [Kock et al., 2010] to arrive at the following moto:

> *"An n-opetope is a tree whose nodes are $(n-1)$-opetopes, and whose edges are $(n-2)$-opetopes."*

Definition (sketch)

Here is how it goes graphically:

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- there is a unique 0-opetope, the *point*, drawn as

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- there is a unique 0-opetope, the *point*, drawn as

$$\cdot$$

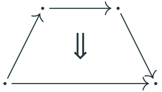
- there is a unique 1-opetope, the *arrow*, drawn as

$$\cdot \longrightarrow \cdot$$

  notice how both ends of the arrow are points (i.e. 0-opetopes);

Definition (sketch)

- a 2-opetope is a shape of the form:



  where the top part (*source*) is any arrangement (or pasting scheme) of 1-opetopes glued along 0-opetopes, and where the bottom part (*target*) consists in **only one** 1-opetope.

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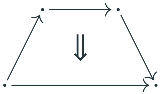
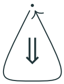
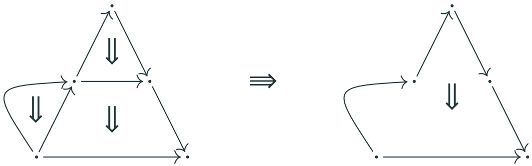


  where the top part (*source*) is any arrangement (or pasting scheme) of 1-opetopes glued along 0-opetopes, and where the bottom part (*target*) consists in **only one** 1-opetope.
  Other examples of 2-opetopes include



Definition (sketch)

- a 3-opetope is a shape of the form:



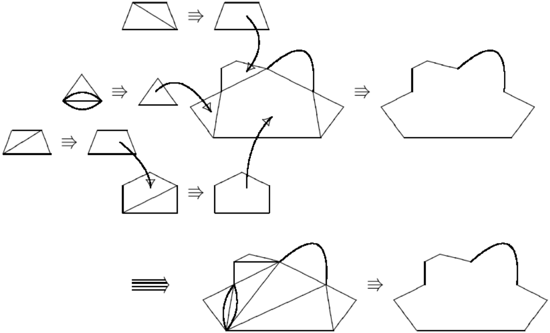
where the left part (*source*) is any pasting scheme of 2-opetopes glued along 1-opetopes, and where the right part (*target*) consists in **only one** 2-opetope **parallel to the overall boundary of the source**.

Definition (sketch)

- and so on: an n-opetope (for $n \geq 2$) is a *source* pasting scheme of $(n-1)$-opetopes glued along $(n-2)$-opetopes, together with a *target* parallel $(n-1)$-opetope.

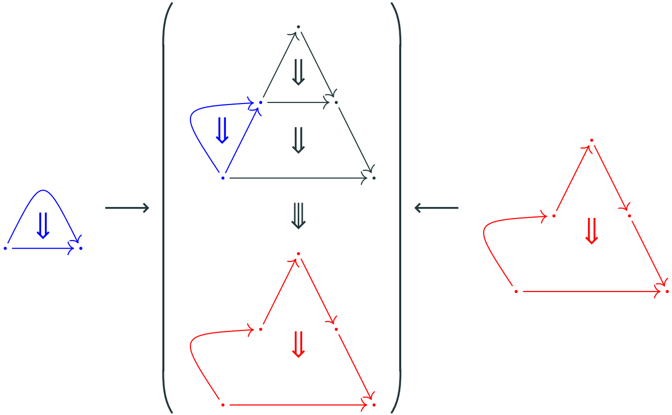
Definition (sketch)

- and so on: an n-opetope (for $n \geq 2$) is a *source* pasting scheme of $(n-1)$-opetopes glued along $(n-2)$-opetopes, together with a *target* parallel $(n-1)$-opetope.
  Here is an example of 4-opetope [Cheng and Lauda, 2004]:



The category of opetopes

There is a very graphical idea of “face of an opetope”:



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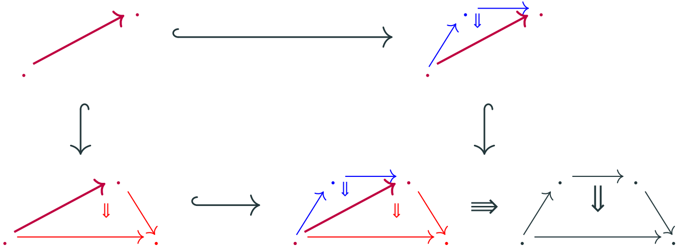
together with 4 relations that implement the geometrical intuition.

Relation [Inner]

The purple 1-face embeds as both the target of the blue 2-face, and a source of the red 2-face. Thus both ways of embedding that 1-face into the whole 3-opetope should be the same.



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The bottom 1-face of the source and the bottom 1-face of the target are geometrically the same, and thus the relevant embeddings should be equal.

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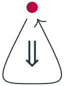
Relation [Glob2]

Likewise, a 1-face in the source of the source is the same as some 1-face in the source of the target, and thus the relevant embeddings should be equal.

Relation [Degen]



In this 2-opetope, the source doesn't contain any 1-face, so that the target is "glued on both ends". The source and the target of the target 1-face are geometrically the same, and thus the relevant embeddings should be equal.

Main result

Statement of the main result

Write $\hat{\mathbb{O}} = [\mathbb{O}^{\mathrm{op}}, \mathcal{S}\mathrm{et}]$ for the category of $\mathcal{S}\mathrm{et}$-valued presheaves over $\mathbb{O}$, aka *opetopic sets*.

Statement of the main result

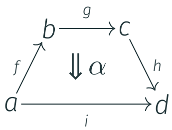
Write $\hat{\mathbb{O}} = [\mathbb{O}^{\mathrm{op}}, \mathcal{S}\mathrm{et}]$ for the category of $\mathcal{S}\mathrm{et}$-valued presheaves over $\mathbb{O}$, aka *opetopic sets*.

Theorem [HT, 2018]

There is an equivalence of categories $\mathcal{P}\mathrm{ol}^{\triangledown} \simeq \hat{\mathbb{O}}$.

Key insight

In an opetopic set Cells are opetopic shapes with labeled faces



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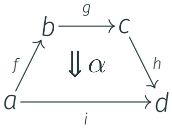

In a many-to-one polygraph Generators are many-to-one, i.e. their source are compositions of (many-to-one) generators, while their target consists in a unique generator:

$$\alpha : hgf \longrightarrow i.$$

Goal

An opetopic set should induce a many-to-one polygraph whose generators are cells.

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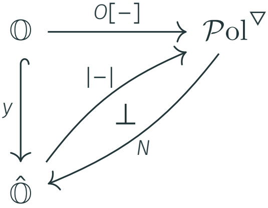
Plan of attack

We construct a Kan “realization–nerve” adjunction, and prove that it is an equivalence:



The opetal functor

The *opetal* functor

$$O[-] : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}^{\nabla}$$

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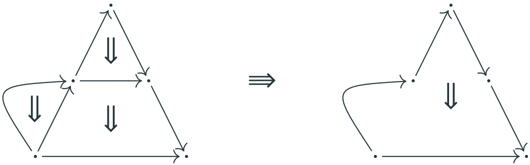
is tricky to construct formally, but the intuition is simple. Given an opetope ω



create a polygraph $O[\omega]$ whose k-generators are the k-faces of ω:

$$O[\omega]_k = \mathbb{O}_k / \omega.$$

The opetal functor

Properties

- By the very nature of opetopes, $O[\omega]$ is many-to-one.

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- For $y\omega \in \hat{\mathbb{O}}$ the representable at ω, we have

$$y\omega_k = \coprod_{\psi \in \mathbb{O}_k} y\omega_\psi = O[\omega]_k.$$

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$$y\omega_k = \coprod_{\psi \in \mathbb{O}_k} y\omega_\psi = O[\omega]_k.$$

- Really, $O[\omega]$ is $y\omega$ with added formal composites of faces of ω.

Polygraphic realization

The left Kan extension of $O[-]$ along y is given by

$$\begin{aligned} |-| = \mathrm{Lan}_y\, O[-] : \hat{\mathbb{O}} &\longrightarrow \mathcal{P}\mathrm{ol}^{\triangledown} \\ X &\longmapsto \operatorname{colim}\left(y/X \longrightarrow \mathbb{O} \xrightarrow{O[-]} \mathcal{P}\mathrm{ol}^{\triangledown} \right). \end{aligned}$$

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From an opetopic set X, it creates a many-to-one polygraph $|X|$ whose n-generators are the n-cells of X, i.e.

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$$|X|_n = \coprod_{\omega \in \mathbb{O}_n} X_\omega .$$

Recall our objective:

"An opetopic set should induce a many-to-one polygraph whose generators are cells."

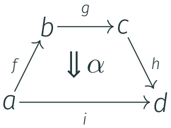
Opetopic nerve

The right adjoint to $|-|$ is given by

$$\begin{aligned} N : \mathcal{P}\mathrm{ol}^{\nabla} &\longmapsto \hat{\mathbb{O}} \\ P &\longmapsto \mathcal{P}\mathrm{ol}^{\nabla}(O[-], P) \end{aligned}$$

Example

If $\alpha \in P_2$, $\alpha : hgf \longrightarrow i$



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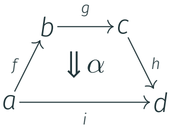
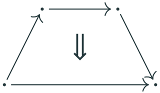


then the shape of α is

$\alpha^\natural =$



so that there is a cell $\alpha \in NP_\omega$, for $\omega = \alpha^\natural$ the opetope above.
Moto: the shape function $(-)^\natural$ "removes labels".

Key result

Theorem ("Yoneda lemma")

For $P \in \mathcal{P}\mathrm{ol}^{\nabla}$ and $x \in P_n$ a generator, there exist a unique pair $\omega \in \mathbb{O}$ and $f : O[\omega] \longrightarrow P$ such that $f(\omega) = x$. Moreover, $\omega = x^{\natural}$.

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For $\omega \in \mathbb{O}_n$, elements of NP_ω are n-generators of P of *shape* ω, and

$$P_n = \coprod_{\omega \in \mathbb{O}_n} NP_\omega .$$

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Recall our objective:

"A many-to-one polygraph should induce an opetopic set whose cells are generators."

Corollary (Main result)

The counit $\varepsilon : |NP| \longrightarrow P$ is a natural isomorphism. After a little more work, we show that $| - | \dashv N$ is an adjoint equivalence of categories.

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Corollary (An open question of [Henry, 2017])

For $\mathtt{1}$ the terminal object of $\mathcal{P}\mathrm{ol}^{\nabla}$, the shape function gives a bijection $(-)^{\natural} : \mathtt{1}_n \longrightarrow \mathbb{O}_n$. Thus opetopes are generators of the terminal many-to-one polygraph.

Conclusion

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- We proved that the category of many-to-one polygraphs $\mathcal{P}\mathrm{ol}^{\triangledown}$ is a presheaf category, and displayed opetopes (in the sense of [Leinster, 2004] and [Kock et al., 2010]) as the adequate shapes.

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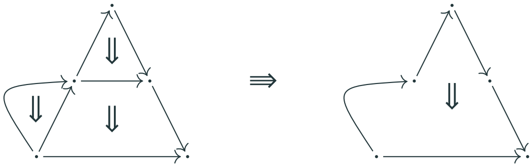
- The main idea was to consider opetopes as describing compositions of lower dimensional opetopes.



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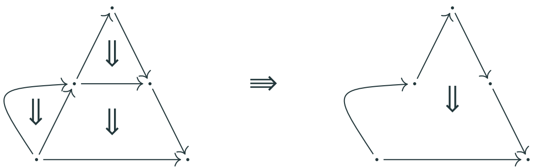

- However, the precise formulations and proofs require the theory of polynomial functors and trees [Gambino and Kock, 2013, Kock, 2011, Kock et al., 2010].

Thank you for your attention!

Baez, J. C. and Dolan, J. (1998).
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