Opetopic algebras

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Journées LHC, October 16th, 2019

[1]IRIF, Paris Diderot University, INSPIRE 2017 Fellow, This project has received funding from the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement No 665850

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This presentation informally presents some of the main notions and results of our upcoming preprint *Opetopic spaces as models for ∞-categories and planar ∞-operads* (on arXiv soon™).

Opetopes

Motivations

Opetopic algebras

Opetopic algebras: monadic approach

The algebraic trompe-l'œil

Opetopes

In a nutshell...

Opetopes are shapes (akin to globules, cubes, simplices, dendrices, etc.) designed to represent the notion of composition in every dimension. As such, they were introduced in [Baez and Dolan, 1998] to describe laws and coherence in weak higher categories.

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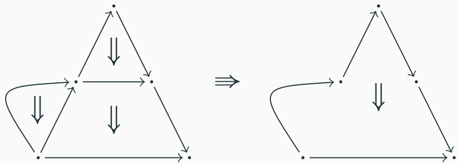
They have been actively studied over the recent years in [Hermida et al., 2002], [Cheng, 2003], [Leinster, 2004], [Kock et al., 2010].

Informal definition

They are **pasting diagrams** where every cell is **many-to-one** i.e. many inputs, one output. Here is an example of a 3-opetope:



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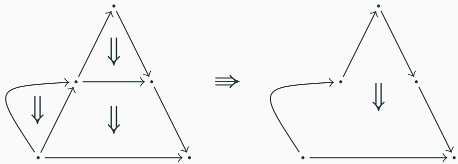

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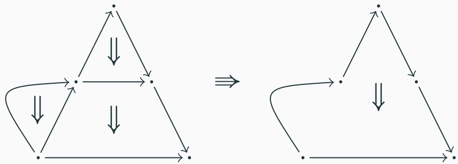

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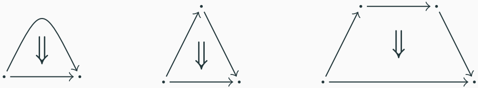
We further ask those cells of dimension 2 **to be** 2**-opetopes**, i.e. pasting diagram of cells of dimension 1 (the simple arrows $\rightarrow$).



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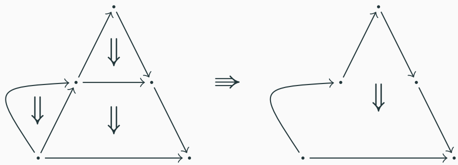

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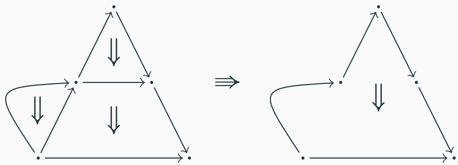

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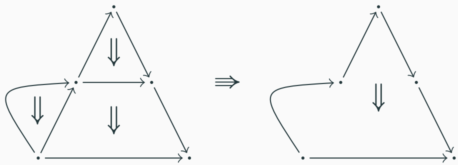

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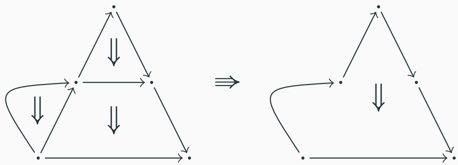

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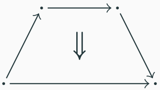

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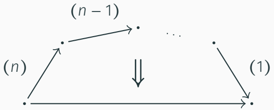

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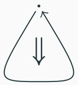

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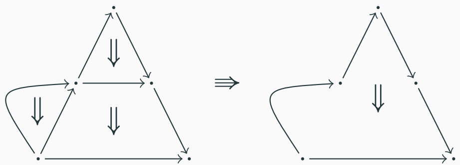

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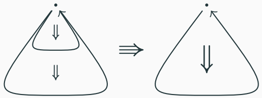

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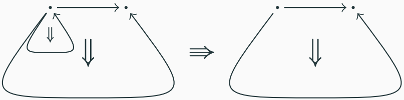
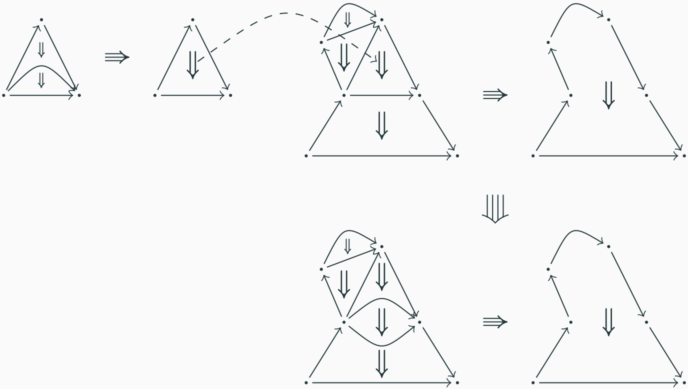


Definition: dimension 4

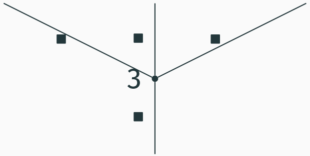
- The induction goes on: 4-opetopes are pasting diagrams of 3-opetopes:

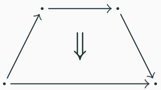


Motivations

Motivations: operads

Let $\mathcal{P}$ be a planar operad. An operation $f \in \mathcal{P}(3)$ is classically represented as a corolla (left), but can also be depicted as 2-opetope (right):





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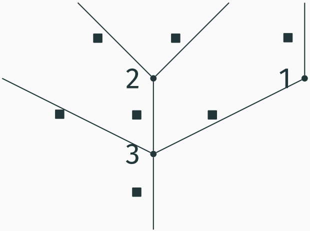
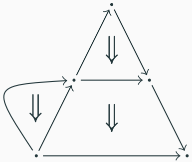
Composing operations of $\mathcal{P}$ amounts to assemble a "tree of operations" (left), which corresponds to forming a pasting diagram (right):





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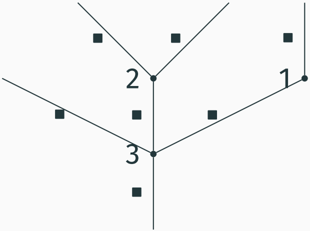
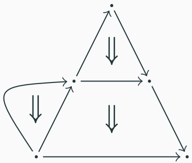
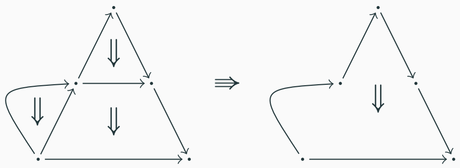




Recall that a pasting diagram of 2-opetopes is a 3-opetope!

Motivations: operads

The associated 3-opetope then corresponds to the *compositor* of this pasting diagram:



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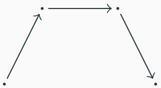

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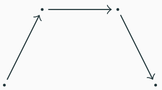
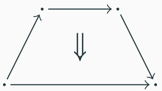


and the *compositor* is the corresponding 2-opetope



Opetopic algebras

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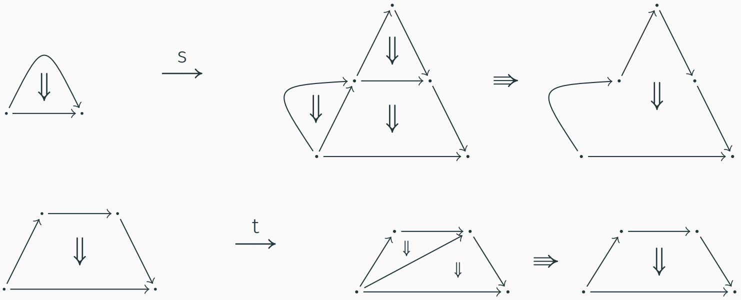

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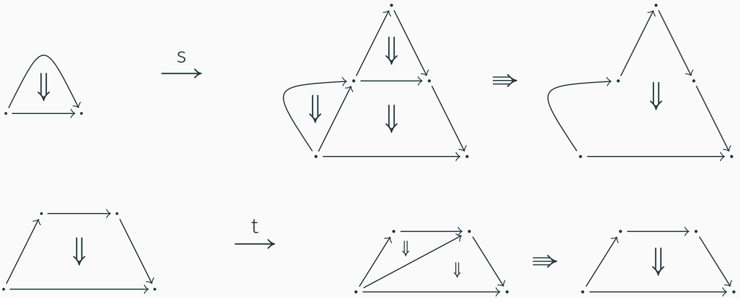

Let $\mathbb{O}_{m,n}$ be the full subcategory of $\mathbb{O}$ spanned by opetopes of dimension between m and n.

Opetopic sets

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Example

1. We have

$$\mathbb{O}_{0,1} = (\blacklozenge \rightrightarrows \blacksquare) \qquad \text{since } \blacksquare = \cdot \longrightarrow \cdot$$

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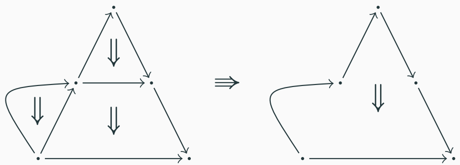
2. Likewise, $\mathcal{P}\mathrm{sh}(\mathbb{O}_{1,2})$ is the category of (non-symmetric) collections.

Some opetopic sets are of particular interest:



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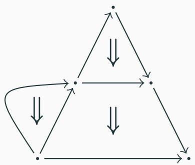

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- Let $\Lambda^{\mathsf{t}}[\omega] = \partial O[\omega] - \{\mathsf{t}\omega\}$ be the *target horn of* ω

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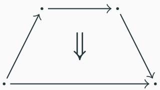
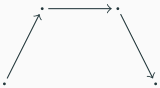
If $\omega = \mathbf{3} =$  , then $\Lambda^{\mathrm{t}}[\mathbf{3}] =$  .

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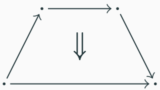
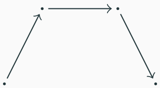
If $\omega = \mathbf{3} =$  , then $\Lambda^{\mathrm{t}}[\mathbf{3}] =$  . Thus, a morphism $\Lambda^{\mathrm{t}}[\mathbf{3}] \longrightarrow X$ amounts to the choice of 3 composable arrows of X.

Lifting against horn inclusions

Lifting $f : \Lambda^{\mathrm{t}}[\omega] \longrightarrow X$ through $O[\omega]$ requires to find a compositor for the pasting diagram of f

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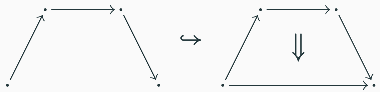
In our previous example,

$\mathrm{h}_\omega :$ 

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An opetopic set $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ such that $\mathsf{H}_{n+1} \perp X$, i.e.

$$\begin{array}{ccc} \Lambda^{\mathsf{t}}[\omega] & \xrightarrow{\forall} & X \\ {\scriptstyle \mathsf{h}_\omega} \downarrow & \nearrow_{\exists!} & \\ O[\omega] & & \end{array}$$

has all compositors of n-dimensional pasting diagrams: **every pasting diagram of dimension n has a composite**.

Lifting against horn inclusions

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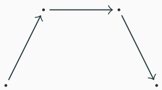

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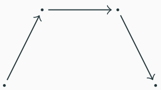

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$$\mu : \text{paths of } X \longrightarrow \text{arrows of } X$$

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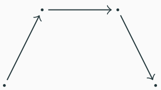

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which looks like a category! But is μ associative?

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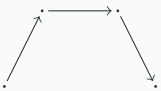
Example

Recall that $\mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1}) = \mathcal{G}\mathrm{raph}$. Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1})$. Pasting diagram of dimension 1 look like this:



If $H_2 \perp X$, then we have a composition map

$$\mu : \text{paths of } X \longrightarrow \text{arrows of } X$$

which looks like a category! But is μ associative? (no)

Lifting against horn inclusions

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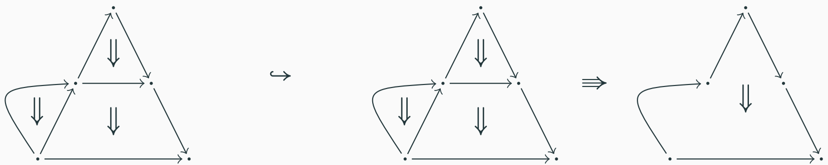
Solution: lift against $\mathsf{H}_{n+1,n+2} = \mathsf{H}_{n+1} \cup \mathsf{H}_{n+2}$.

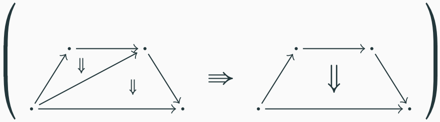
Intuitively, if $\mathsf{H}_{n+2} \perp X$, then a combination of lifting problems (in dimension n) can be summarized into a unique one:



Lifting against horn inclusion

Example

Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ be an opetopic set such that $\mathsf{H}_{2,3} \perp X$, and consider

$\omega =$  $\in \mathbb{O}_3$

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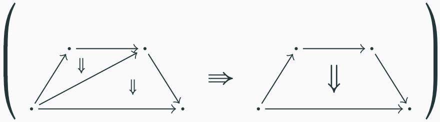
$\omega =$  $\in \mathbb{O}_3$

Then $\mathsf{h}_\omega \perp X$ ensures that for f, g, h composable arrows in X we have

$$(fg)h = fgh.$$

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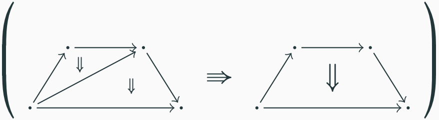
$\omega =$  $\in \mathbb{O}_3$

Then $\mathsf{h}_\omega \perp X$ ensures that for f, g, h composable arrows in X we have

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A similar opetope would enforce $f(gh) = fgh$.

Opetopic algebras (almost)

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So to summarize:

- $\mathsf{H}_{n+1} \perp X$ gives a composition operation for n-dimensional cells of X;
- $\mathsf{H}_{n+2} \perp X$ ensures that it is suitably associative.

The last step required to define opetopic algebra is to trivialize X in dimension $< n$ and $> n+2$.

Trivialization

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Lemma

$$\mathrm{H}_{n+1,n+2} \cup \mathrm{B}_{>n+2} \perp X \quad \Longleftrightarrow \quad \mathrm{H}_{\geq n+1} \perp X$$

Opetopic algebras

Definition

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- Loday's combinads (over the combinatorial pattern $\mathbb{PT}$ of planar trees) are exactly $(0, 3)$-opetopic algebras.

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Opetopic algebras: monadic approach

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We now describe the "free (k, n)-algebra"-monad, which constructs all those pasting diagrams.

The $\mathfrak{Z}^n$ monad

Discarding irrelevant dimensions, we want a monad $\mathfrak{Z}^n : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ that "constructs pasting diagrams".

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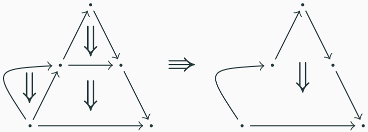

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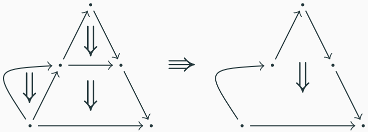
- Let $\omega \in \mathbb{O}_{n+1}$ be  A pasting diagram as on the left ($\Lambda^{t}[\omega]$) needs to be evaluated to a cell as on the right ($\mathrm{t}\omega$).

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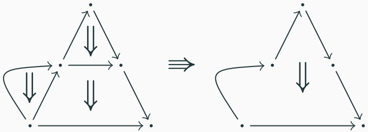

  A pasting diagram as on the left ($\Lambda^{\mathsf{t}}[\omega]$) needs to be evaluated to a cell as on the right ($\mathsf{t}\omega$). Thus for $\psi \in \mathbb{O}_n$,

$$\mathfrak{Z}^n Y_\psi = \sum_{\substack{\omega \in \mathbb{O}_{n+1} \\ \mathsf{t}\omega = \psi}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \left(\Lambda^{\mathsf{t}}[\omega], Y \right).$$

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- Unit $Y \longrightarrow \mathfrak{Z}^n Y$: a single cell of Y is already a pasting diagram.

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We write $\mathcal{A}\mathrm{lg}^k(\mathfrak{Z}^n)$ the Eilenberg–Moore category of $\mathfrak{Z}^n : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$.

Opetopic algebras: monadic definition

Recall that (k, n)-opetopic algebras are opetopic sets $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ such that $\mathsf{A}_{k,n} \perp X$.

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There is an adjunction

$$h_{k,n} : \mathcal{P}\mathrm{sh}(\mathbb{O}) \rightleftarrows \mathcal{A}\mathrm{lg}^k(\mathfrak{Z}^n) : N_{k,n}$$

that exhibits $\mathcal{A}\mathrm{lg}^k(\mathfrak{Z}^n)$ as the localization $\mathsf{A}_{k,n}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{O})$. In other words, (k, n)-algebras and $\mathfrak{Z}^n$ algebras are the same!

Opetopic algebras: monadic definition

Examples

- If $(k, n) = (0, 1)$, then $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) = \mathcal{S}\mathrm{et}$, and $\mathfrak{Z}^1 : \mathcal{S}\mathrm{et} \longrightarrow \mathcal{S}\mathrm{et}$ is the free monoid monad.

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- If $(k, n) = (1, 2)$, then $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) = \mathcal{C}\mathrm{oll}$ is the category of (non symmetric) collections, and $\mathfrak{Z}^2 : \mathcal{C}\mathrm{oll} \longrightarrow \mathcal{C}\mathrm{oll}$ is the free colored planar operad monad.

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- If $(k, n) = (0, 1)$, then $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) = \mathcal{S}\mathrm{et}$, and $\mathfrak{Z}^1 : \mathcal{S}\mathrm{et} \longrightarrow \mathcal{S}\mathrm{et}$ is the free monoid monad.
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So we have an infinite hierarchy of “higher arity algebras”!

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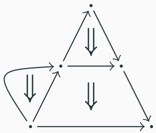
The algebraic trompe-l’œil

Too many colors

Recall that a n-dimensional pasting diagram in X is a set of n-cells of X glued along $(n-1)$-cells



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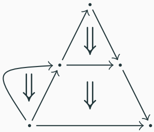

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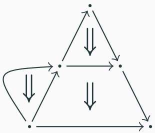

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Theorem

The following is a pullback

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}^k(\mathfrak{Z}^n) & \longrightarrow & \mathcal{A}\mathrm{lg}^1(\mathfrak{Z}^n) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) & \longrightarrow & \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}). \end{array}$$

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Too many everything

Pasting the two pullbacks

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}^k(\mathfrak{Z}^n) & \longrightarrow & \mathcal{A}\mathrm{lg}^1(\mathfrak{Z}^n) \\ {\scriptstyle U}\downarrow & \lrcorner & \downarrow{\scriptstyle U} \\ \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) & \longrightarrow & \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}) \end{array}, \qquad \begin{array}{ccc} \mathcal{A}\mathrm{lg}^1(\mathfrak{Z}^n) & \longrightarrow & \mathcal{A}\mathrm{lg}^1(\mathfrak{Z}^3) \\ {\scriptstyle U}\downarrow & \lrcorner & \downarrow{\scriptstyle U} \\ \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}) & \xrightarrow{(-)_\dagger} & \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3}) \end{array}.$$

we obtain

Theorem (Algebraic trompe-l'œil)

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Thank you for your attention!

Stay tuned for part 2 with Chaitanya!

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