# Opetopes

## Syntactic and Algebraic Aspects

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Under the supervision of Pierre-Louis Curien[1] and Samuel Mimram[2]

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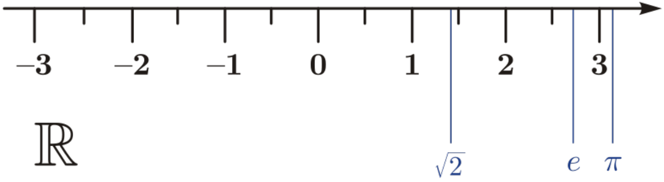
# Introduction

# Des nombres avec plus de dimensions

Considérons la droite des nombres réels :



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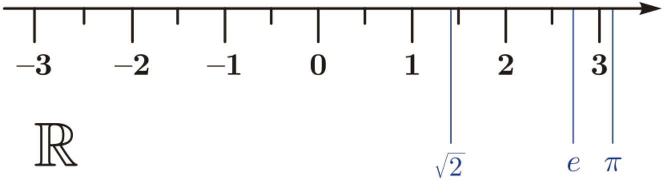


Elle contient les nombres réels, qui servent à quantifier quasiment tout ce qui est… réel : le volume d'essence dans le réservoir d'une voiture, la masse du soleil, le PIB, l'aire d'un polygone, etc.

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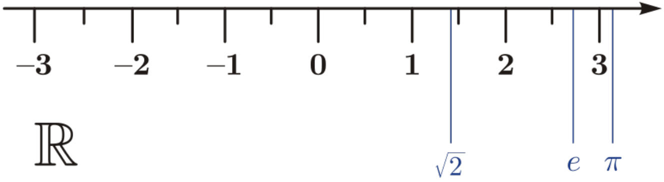


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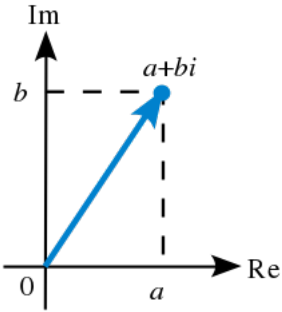
Une droite est un objet à 1 dimension. On peut donc dire que les nombres réels sont des “nombres à 1 dimension”.

# Des nombres avec plus de dimensions

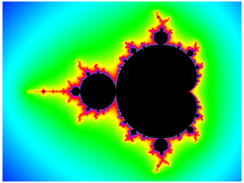
A quoi ressemblerait un nombre à 2 dimensions ?

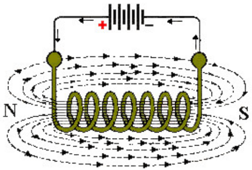
# Des nombres avec plus de dimensions

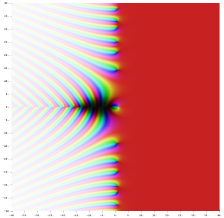
A quoi ressemblerait un nombre à 2 dimensions ? C'est précisément ce que l'on appelle un *nombre complexe*.

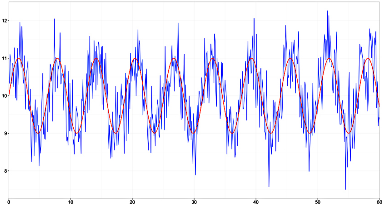


Ils ne quantifient pas nécessairement quelque chose de réel, mais ils trouvent des applications en géométrie, en physique, en théorie des nombres, en traitement de signaux, etc.



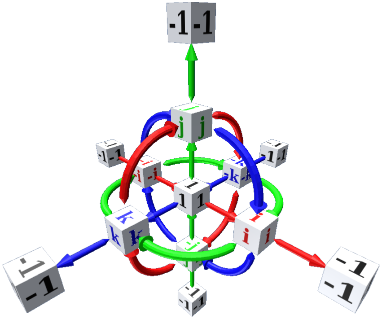




# Des nombres avec plus de dimensions

Mais on peut aller encore plus loin : un nombre à 4 dimensions s'appelle un *quaternion*.



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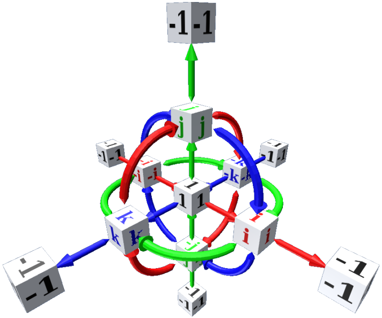

Ils sont beaucoup utilisés informatique (e.g. les moteurs 3D), et en physique.

# Des nombres avec plus de dimensions

Il existe même des nombres à 8 dimensions (*octonions*) et à 16 dimensions (*sédénions*).

Ils sont également utilisés en physique théorique et dans certaines branches des mathématiques. De récents travaux leur ont également trouvé des applications en robotique.

# La physique avec plus de dimensions

Cette idée d'ajouter toujours plus de dimensions se retrouve également en physique.

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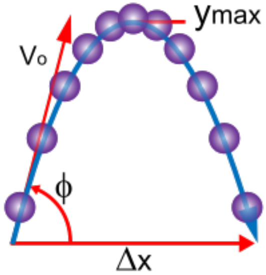
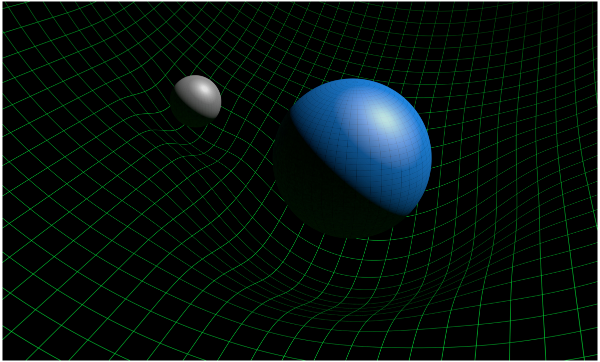
Mécanique newtonienne : 3 dimensions



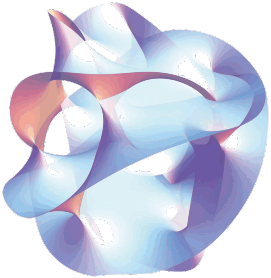
# La physique avec plus de dimensions

Théorie de la relativité : 4 dimensions

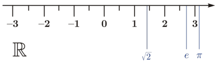


# La physique avec plus de dimensions

Théorie des cordes : 10 dimensions (ou 11, ou 26, suivant à qui vous demandez)



Moralité : plus de dimensions = plus de possibilités !

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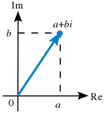


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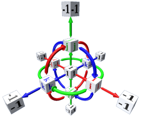

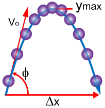


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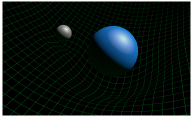

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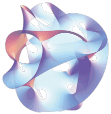

# Les maths avec plus de dimensions

Depuis quelques dizaines d’années, les mathématiciens essayent d’ajouter des dimensions aux mathématiques elles-mêmes.

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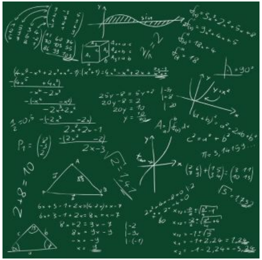
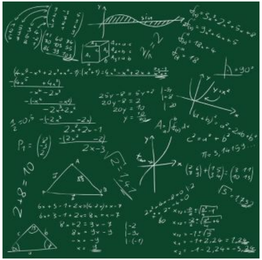
Mais qu’est-ce que ca veut dire, des “mathématiques avec plus de dimensions” ?



$\rightsquigarrow$ ???

# Les maths avec plus de dimensions

Une des réponses à cette question est la *théorie des catégories supérieures* (en anglais, *higher dimensional category theory*, ou juste *higher category theory*).



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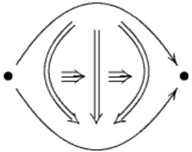
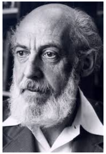
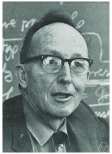


# La théorie des catégories

La *théorie des catégories* classique est une branche des mathématiques née en 1942, et se propose comme une approche synthétique aux mathématiques elles-mêmes.

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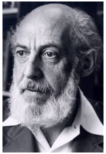
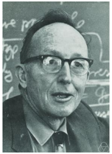
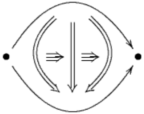




La *théorie des catégories supérieures* est née dans les années 60-70, et pousse certains principes de la théorie des catégories classique beaucoup plus loin.



# La théorie des catégories supérieures

Analogie : Si l'on imagine une théorie mathématique comme une maison, alors elle est construite du bas vers le haut, chaque théorème étant une brique, reposant sur d'autres briques plus basses.

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En théorie des catégories supérieures, on a certains étages, mais pas les premiers malheureusement...

# Fondations

La première étape pour construire ces étages manquants est de choisir les briques.

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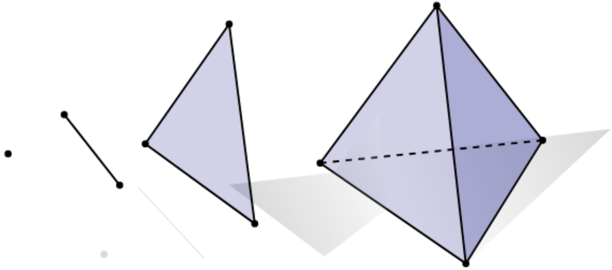
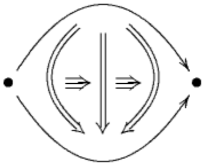
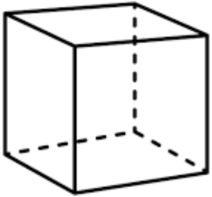
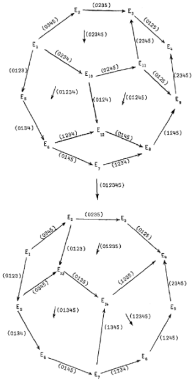
De nos jours, les briques les plus utilisées sont les *simplexes*



Parmis les autres modèles célèbres, on retrouve les *globules*, les *cubes*, ou encore les *orientaux* :

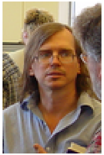


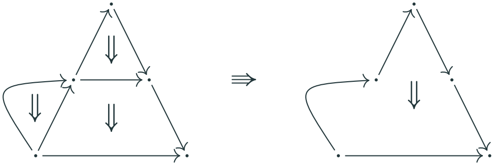




# Opétopes

Dans cette thèse, nous nous intéressons à un modèle de briques moins utilisé : les *opétopes*, créés par John Baez et James Dolan en 1998.



1. *Opétopes* : On détaille la spécification et la composition exacte de cette fameuse “brique opétopique”. On construit également les outils de chantier nécessaires à la bonne utilisation et au bon assemblage de ces briques.

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3. *Algèbres* : On montre que certains bâtiments célèbres peuvent être reconstruits à l’identique en utilisant ces briques opétopiques. De par la nature ces briques, on pourrait même ajouter des étages.

Now we switch to English.

# Opetopes

# In a nutshell...

Opetopes are shapes (akin to globes, cubes, simplices, dendrices, etc.) introduced in [Baez and Dolan, 1998] to describe laws and coherence in weak higher categories.

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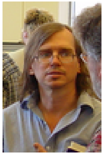
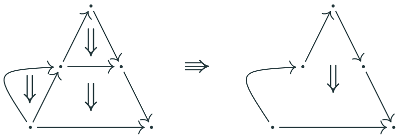


Basically, they represent the notion of composition in every dimension. They have been actively studied in the first decade of this century, e.g. [Hermida et al., 2002], [Cheng, 2003], [Leinster, 2004], [Kock et al., 2010].

# Informal definition

Opetopes are **pasting diagrams** where every cell is **many-to-one** i.e. many inputs, one output. Here is an example of a 3-opetope:



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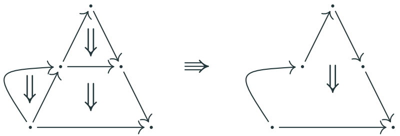

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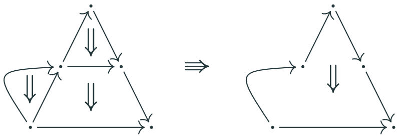

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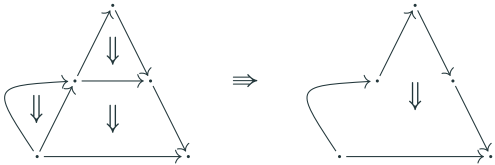
**We further ask these cells of dimension** 2 **to be** 2**-opetopes**, i.e. pasting diagrams of cells of dimension 1 (the simple arrows $\rightarrow$).

# Informal definition



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An $n$-**dimensional opetope** (or just $n$-**opetope**) is a pasting diagram of $(n-1)$-opetopes,

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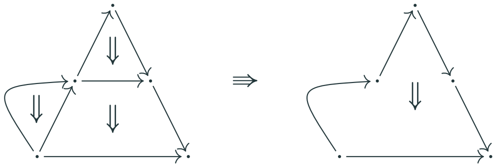

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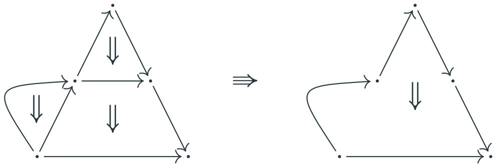

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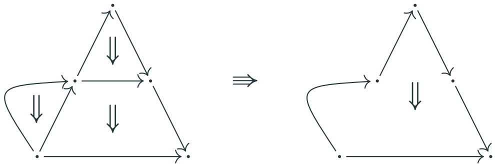

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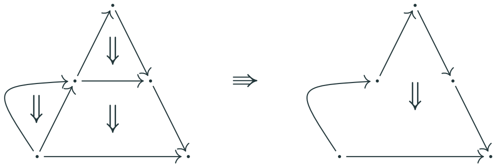

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We write $\mathbb{O}_n$ for the set of $n$-opetopes, and $\mathbb{O}$ for the set of all opetopes.

# Informal definition: low dimensions

- There is a unique 0-dimensional opetope, which we'll call the **point** and denote by $\blacklozenge$:

$$\cdot$$

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- 2-opetopes are pasting diagrams of 1-opetopes:

$$\mathbf{3} \qquad = \qquad \begin{tikzcd} & \cdot \arrow[r] & \cdot \arrow[dr] & \\ \cdot \arrow[ur] \arrow[rrr] & & \arrow[ul, phantom, "\Downarrow"] & \cdot \end{tikzcd}$$

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$$\mathbf{n} \quad = \quad \begin{tikzcd} & \cdot \arrow[r, "(n-1)"] & \cdot & \cdots & \cdot \arrow[dr, "(1)"] & \\ \cdot \arrow[ur, "(n)"] \arrow[rrrrr] & & & \Downarrow & & \cdot \end{tikzcd}$$

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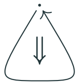
$0 \quad = \quad$ 

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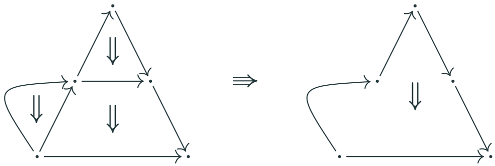

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$$
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\cdot \arrow[ur] \arrow[urr] \arrow[rrr] & & & \cdot
\end{tikzcd}
\quad \Rrightarrow \quad
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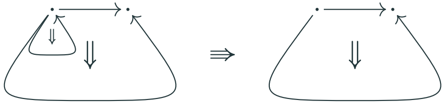
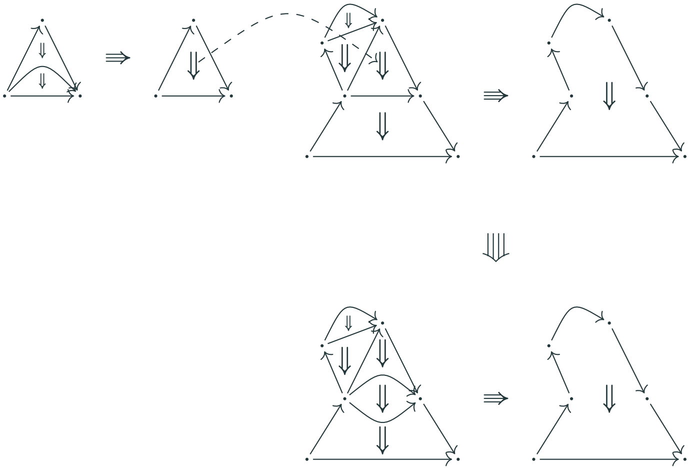


# Informal definition: dimension 4

- The induction goes on: 4-opetopes are pasting diagrams of 3-opetopes:



# Formal definition

As expected, opetopes are not easy to define formally.

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As expected, opetopes are not easy to define formally. The idea of KJBM [Kock et al., 2010] is to take the “tree & pasting diagram” duality very seriously.



# Polynomial functors, trees, monads

The notion of tree is supported by the theory of *polynomial functors*.

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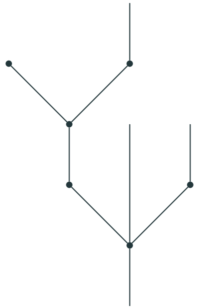
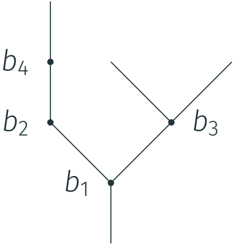
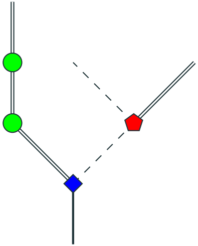
- the set of *nodes* (or *operations*) is $B$;
- the set of input and output *edges* (or *colors*) is $I$;
- for a given node $b \in B$, the inputs of $b$ are parametrized by $p^{-1}(b)$, and its output is $t(b)$.

A polynomial functor $T$ is a *polynomial tree* if it is possible to label it with its operations and colors.



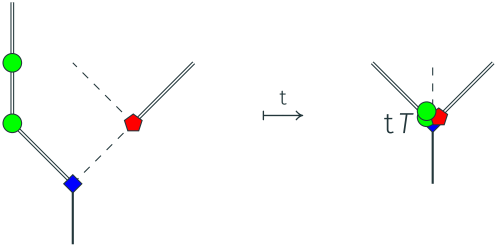
If $T$ is polynomial tree, and $P$ is a polynomial functor, then a morphism $\phi : T \longrightarrow P$ *decorates* the nodes of $T$ by operations of $P$, and the edges of $T$ by colors of $P$.



We call $T$ (leaving $\phi$ implicit) a $P$-tree, and write $\operatorname{tr} P$ for the category of $P$-trees.

If $P$ is a *polynomial monad*, then we can compute the *target* of a $P$-tree $T$ by "contracting" it, yielding an operation of $P$:



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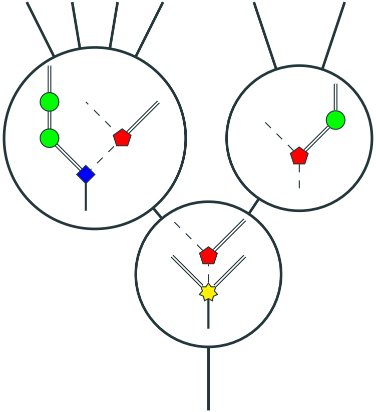
the *Baez–Dolan construction* $(-)^+$ creates a new polynomial monad $P^+$ whose operations are precisely the $P$-trees:

$$B \xleftarrow{\mathrm{s}} \mathrm{tr}^\bullet P \xrightarrow{p} \mathrm{tr}\, P \xrightarrow{\mathrm{t}} B$$

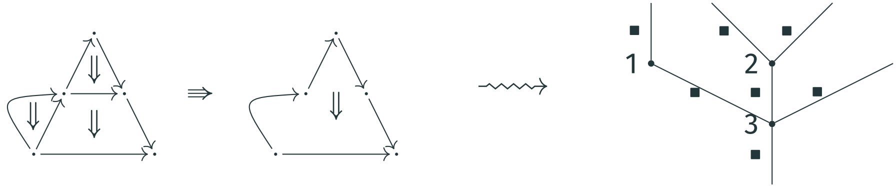
# The Baez–Dolan construction

Thus, a $P^+$-tree is a tree decorated by trees decorated by operations of $P^+$, i.e. a tree of $P$-trees:



# Formal definition

Why is all that relevant? If a pasting diagram is a tree,



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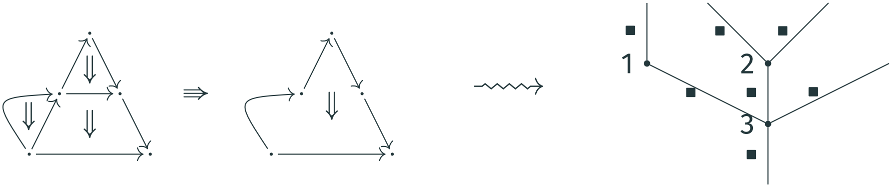


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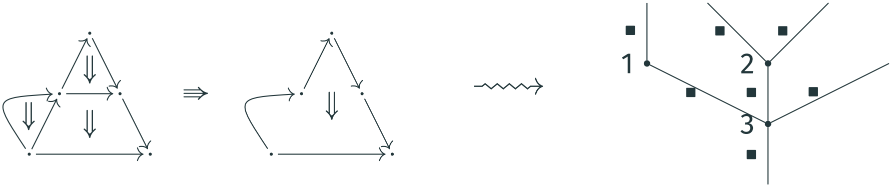
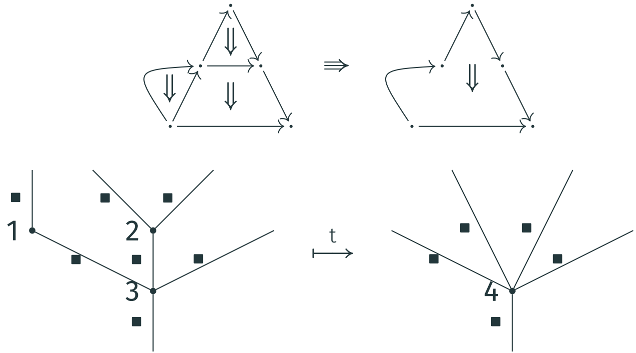



then an opetope is a tree of of opetopes, i.e. a

$$\underbrace{\text{tree of trees of trees of trees of } \ldots \text{ of trees of}}_{n \text{ times}} \text{ points.}$$

Furthermore, the “contraction” operation corresponds exactly to targets:



# Formal definition

We can finally define opetopes properly. Start with the identity polynomial monad $\mathfrak{Z}^0 := \mathsf{id}_{\mathcal{S}\mathrm{et}}$, and write it as

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**Definition**

An $n$-opetope is a color of $\mathfrak{Z}^n$. If $n \geq 1$, it is equivalently an operation of $\mathfrak{Z}^{n-1}$. If $n \geq 2$, it is equivalently a $\mathfrak{Z}^{n-2}$-tree.

## Formal definition

Unfolding the definition,

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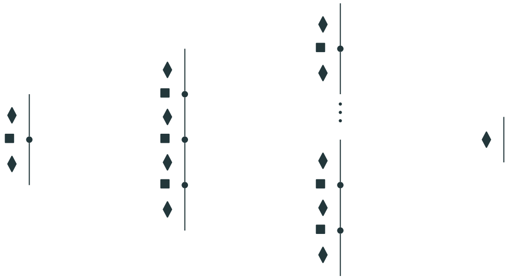
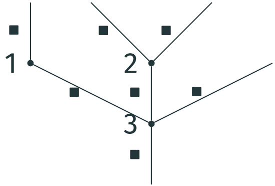
- ⧫ is the only 0-opetope.
- ■ is the only 1-opetope.
- A 2-opetope is a $\mathfrak{Z}^0$-tree. Since $\mathfrak{Z}^0$ has only one operation, which has only one input, a $\mathfrak{Z}^0$-tree is necessarily a linear tree.



We have a correspondence $\mathbb{O}_2 \cong \mathbb{N}$.

# Formal definition

- A 3-opetope is a $\mathfrak{Z}^1$-tree. The operations of $\mathfrak{Z}^1 = (\mathfrak{Z}^0)^+$ are linear trees. Thus, a $\mathfrak{Z}^1$-tree is simply a tree.



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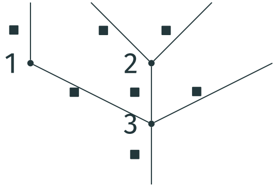


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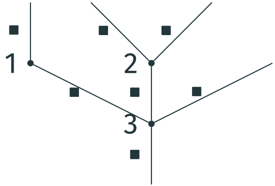


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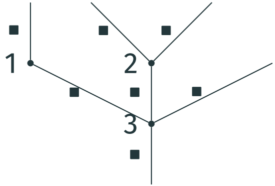


- A 4-opetope is a $\mathfrak{Z}^2$-tree, i.e. a tree of trees.
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- A $(n+1)$-opetope is a tree of $n$-opetopes.

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To summarize:

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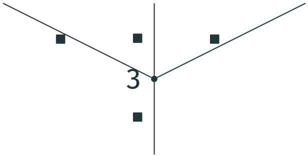
3. $n$-opetopes are the colors of $\mathfrak{Z}^n := (\mathfrak{Z}^0)^{++\cdots+}$

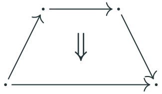
$$\mathbb{O}_n \xleftarrow{\mathsf{s}} E \longrightarrow \mathbb{O}_{n+1} \xrightarrow{\mathsf{t}} \mathbb{O}_n.$$

# Opetopic algebras

# Motivations: operads

Let $P$ be a planar operad. An operation $f \in P_3$ is classically represented as a corolla (left), but can also be depicted as 2-opetope (right):



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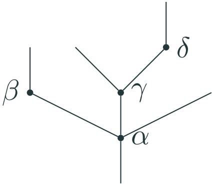
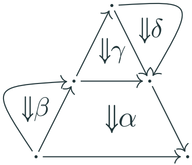
Composing operations of $P$ amounts to assemble a “tree of operations” (left), which corresponds to forming a pasting diagram (right):



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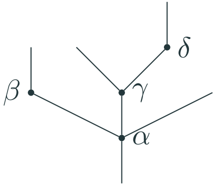
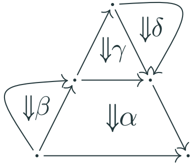
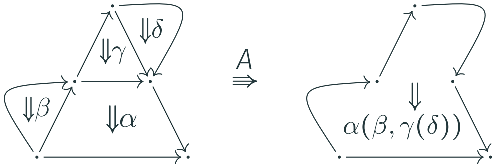



Recall that a pasting diagram of 2-opetopes is a 3-opetope!

# Motivations: operads

The associated 3-opetope then corresponds to the *compositor* of this pasting diagram:



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Categories can also be represented “opetopically”: a morphism in a category $\mathcal{C}$ has the shape of the arrow, which is the unique 1-dimensional opetope:

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and the *compositor* is the corresponding 2-opetope

$$\begin{tikzcd}
 & b \arrow[r, "g"] & c \arrow[r, "h"] \arrow[d, phantom, "\Downarrow\mu"] & d \arrow[dr, "i"] & \\
a \arrow[ur, "f"] \arrow[rrrr, "i \cdot h \cdot g \cdot f"'] & & & & e
\end{tikzcd}$$

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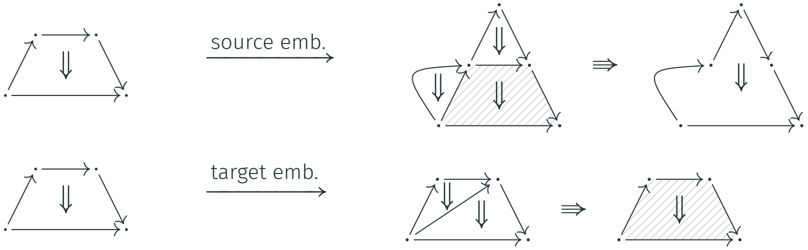

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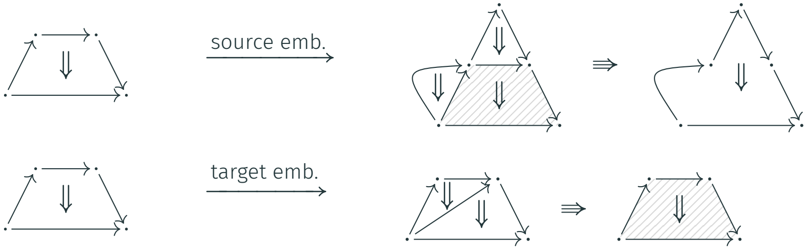

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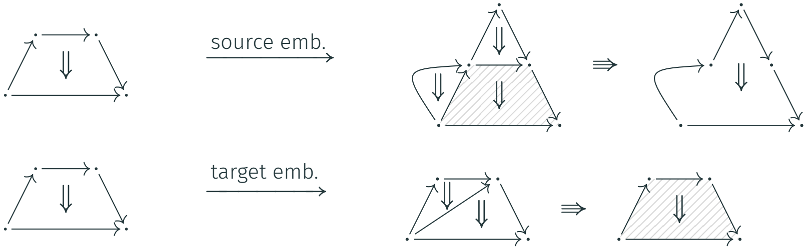


We use the dedicated formalism of *higher addresses* to precisely identify occurrences of cells in pasting diagrams.

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The green arrow has two embeddings into the global 3-opetope:

$$\begin{tikzcd} \blacksquare \arrow[r, "target"] \arrow[d, "source"'] & 2 \arrow[d, "source"] \\ 2 \arrow[r, "source"] & \omega \end{tikzcd}$$

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For example:

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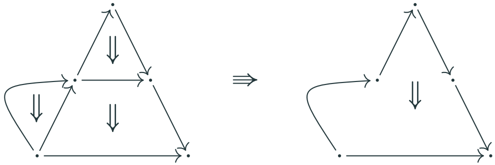
2. Likewise, $\mathcal{P}\mathrm{sh}(\mathbb{O}_{1,2})$ is the category of (non-symmetric) collections.

# Opetopic sets

Some opetopic sets are of particular interest:



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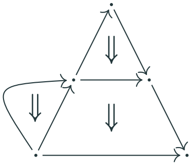

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- Let $S[\omega] = \partial O[\omega] - \{\mathsf{t}\omega\}$ be the *spine of* $\omega$.

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For example, if $\omega = \mathbf{3} = \begin{tikzcd} & \cdot \arrow[r] \arrow[d, phantom, "\Downarrow"] & \cdot \arrow[dr] & \\ \cdot \arrow[ur] \arrow[rrr] & {} & & \cdot \end{tikzcd}$, then $S[\mathbf{3}] = \begin{tikzcd} & \cdot \arrow[r] & \cdot \arrow[dr] & \\ \cdot \arrow[ur] & & & \cdot \end{tikzcd}$ .

# Lifting against spine inclusions

Let $\omega \in \mathbb{O}$, and $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$. A morphism

$$f : S[\omega] \longrightarrow X$$

amounts to forming a pasting diagram of shape $\omega$ with elements of $X$.

For example, if $\omega = \mathbf{3} =$ $\begin{tikzcd}[sep=small] & \cdot \arrow[r] \arrow[dr, phantom, "\Downarrow"] & \cdot \arrow[dr] & \\ \cdot \arrow[ur] \arrow[rrr] & & & \cdot \end{tikzcd}$, then $S[\mathbf{3}] =$ $\begin{tikzcd}[sep=small] & \cdot \arrow[r] & \cdot \arrow[dr] & \\ \cdot \arrow[ur] & & & \cdot \end{tikzcd}$. Thus, a morphism $S[\mathbf{3}] \longrightarrow X$ amounts to the choice of 3 composable arrows of $X$.

# Lifting against spine inclusions

Lifting $f : S[\omega] \longrightarrow X$ through $O[\omega]$ requires to find a *compositor* for the pasting diagram defined by $f$

$$\begin{tikzcd} S[\omega] \arrow[r, "f"] \arrow[d, "s_\omega"'] & X \\ O[\omega] \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

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In our previous example,

$$s_\omega : \quad \begin{tikzcd}[column sep=small] & \cdot \arrow[r] & \cdot \arrow[dr] & \\ \cdot \arrow[ur] & & & \cdot \end{tikzcd} \quad \hookrightarrow \quad \begin{tikzcd}[column sep=small] & \cdot \arrow[r] \arrow[dr, phantom, "\Downarrow"] & \cdot \arrow[dr] & \\ \cdot \arrow[ur] \arrow[rrr] & & & \cdot \end{tikzcd}$$

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# Lifting against spine inclusions

Let $\mathsf{S}_n = \{\mathsf{s}_\omega : S[\omega] \hookrightarrow O[\omega] \mid \omega \in \mathbb{O}_n\}$.

An opetopic set $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ such that $\mathsf{S}_{n+1} \perp X$, i.e.

$$\begin{tikzcd} S[\omega] \arrow[r, "\forall"] \arrow[d, "\mathsf{s}_\omega"'] & X \\ O[\omega] \arrow[ur, "\exists!"'] & \end{tikzcd}$$

has all compositors of $n$-dimensional pasting diagrams: **every pasting diagram of dimension $n$ has a unique composite**.

# Lifting against spine inclusions

For example, recall that $\mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1}) = \mathcal{G}\mathrm{raph}$. Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1})$.

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which looks like a category! But is $\mu$ associative? (nope)

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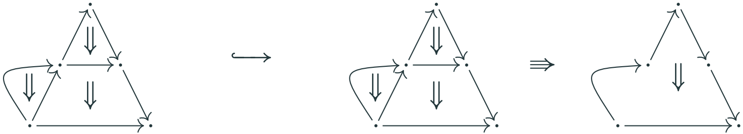
# Lifting against spine inclusions

Unfortunately, lifting against $\mathsf{S}_{n+1}$ does not give an adequate notion of algebra as the composition operation need not be associative.

Solution: lift against $\mathsf{S}_{n+1,n+2} = \mathsf{S}_{n+1} \cup \mathsf{S}_{n+2}$.

Intuitively, if $\mathsf{S}_{n+2} \perp X$, then a combination of lifting problems (in dimension $n$) can be summarized into a unique one:



For example, let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ be an opetopic set such that $\mathsf{S}_{2,3} \perp X$, and consider

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A similar opetope would enforce $f(gh) = fgh$.

So to summarize:

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The last step required to define opetopic algebra is to trivialize $X$ in dimension $< n$ and $> n + 2$.

# Opetopic algebras

- We want $X$ to be "trivial" in dimension $< n$. Solution: require $\mathsf{O}_{<n} \perp X$, where

$$\mathsf{O}_{<n} = \{\varnothing \hookrightarrow O[\psi] \mid \dim \psi < n\}.$$

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**Lemma**

$$\mathsf{S}_{n+1,n+2} \cup \mathsf{B}_{>n+2} \perp X \qquad \Longleftrightarrow \qquad \mathsf{S}_{\geq n+1} \perp X$$

**Definition**

A $(0, n)$-*opetopic algebra* is an opetopic set $X$ such that

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- Monoids are exactly $(0, 1)$-opetopic algebras.
- Planar uncolored operads are exactly $(0, 2)$-opetopic algebras.
- Loday's combinads (over the combinatorial pattern $\mathbb{PT}$ of planar trees) are exactly $(0, 3)$-opetopic algebras.

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- Categories (colored monoids) are exactly $(1, 1)$-opetopic algebras.
- Planar colored operads are exactly $(1, 2)$-opetopic algebras.

Let $\mathcal{A}\mathrm{lg}_{k,n}$ be the category of $(k, n)$-algebras.

**Theorem**

We have a reflective adjunction

$$h : \mathcal{P}\mathrm{sh}(\mathbb{O}) \underset{\perp}{\rightleftarrows} \mathcal{A}\mathrm{lg}_{k,n} : M.$$

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**Corollary**

$\mathcal{A}\mathrm{lg}_{k,n}$ is locally finitely presentable.

# Summary

We presented the notion of opetopic algebras, which encompasses major known kinds of algebraic structures:

| | | | | | |
|---|---|---|---|---|---|
| Algebraic stucture | Sets =0-algebras | Categories =1-algebras | Operads =2-algebras | $\mathbb{PT}$−combinads =3-algebras | … |
| Arity of composition | Trivial =1-opetopes | Lists =2-opetopes | Trees =3-opetopes | Trees of trees =4-opetopes | … |

# Opetopic homotopy theory

In this last part, we show that some results regarding the homotopy theory of these algebras (and their weak counterparts) [Rezk, 2001] [Joyal and Tierney, 2007] [Cisinski and Moerdijk, 2013] [Horel, 2015] can be unified and generalized in the framework of opetopic algebras.

# The category $\mathbb{\Lambda}$

The first step is to find an adequate *shape category* for opetopic algebras.

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It is well known that $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ is a better category to deal with ($\infty$-)categories, and $\mathcal{P}\mathrm{sh}(\mathbb{\Omega})$ for ($\infty$)-operads.

It turns out that $\mathbb{O}$ is not suitable to investigate weak opetopic algebras. For example, it does not generalize $\mathbb{\Delta}$ or $\mathbb{\Omega}$.

# The category $\mathbb{\Lambda}$

Fix $n \geq 1$ and $k = 1$. Similar to categories and operads, there is a small category $\mathbb{\Lambda} = \mathbb{\Lambda}_{1,n}$ such that $(1, n)$-algebras are presheaves over $\mathbb{\Lambda}$ satisfying some lifting condition.

**Theorem**

There is a reflective adjunction

$$\tau : \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) \underset{\perp}{\rightleftarrows} \mathcal{A}\mathrm{lg} : N,$$

where $\mathbb{\Lambda}$ is the full subcategory of $\mathcal{A}\mathrm{lg} = \mathcal{A}\mathrm{lg}_{1,n}$ of algebras of the form $h\omega$, for $\omega \in \mathbb{O}_{n+1}$.

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- $\mathbb{\Delta} = \mathbb{\Lambda}_{1,1}$, as categories are exactly $(1,1)$-algebras, and simplices are exactly free categories on finite linear graphs ($\simeq$ 2-opetopes).

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- $\mathbb{\Omega} = \mathbb{\Lambda}_{1,2}$, as operads are exactly $(1,2)$-algebras, and dendrices are exactly free operads on trees ($\simeq$ 3-opetopes).

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$$\mathsf{S} = \left\{ \mathsf{s}_\omega : S[h\omega] \hookrightarrow h\omega \mid \omega \in \mathbb{O}_{n-k,n} \right\};$$

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**Theorem**

In the reflective adjunction $\tau : \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) \rightleftarrows \mathcal{A}\mathrm{lg} : N$, $\tau$ is the localization at $\mathsf{S}$.

- there is a good notion of *horn* and *inner horn inclusion*:

$$\mathsf{H}_{\mathrm{inner}} = \left\{ \mathsf{h}^{e}_{\omega} : \Lambda^{e}[h\omega] \hookrightarrow h\omega \mid \omega \in \mathbb{O}_{n-k,n}, e \text{ inner face of } h\omega \right\};$$

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- there is a good notion of *interval objects*, parametrized by $\mathbb{O}_{n-1}$:

$$\mathsf{R} = \left\{ \mathsf{r}_\phi : \mathfrak{I}_\phi \twoheadrightarrow h\phi \mid \phi \in \mathbb{O}_{n-1} \right\}$$

(for example, in the simplicial case $(n = 1)$ $\mathsf{r}_\blacklozenge : (* \leftrightarrow *) \twoheadrightarrow \Delta[0]$).

**Theorem**

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Let us denote it by $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})_\infty$. If $n = 1$, $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})_\infty = \mathcal{P}\mathrm{sh}(\mathbb{\Delta})_{\mathrm{Joyal}}$. If $n = 2$, $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})_\infty = \mathcal{P}\mathrm{sh}(\mathbb{\Omega})_{\mathrm{CM}}$.

# Homotopy-coherent opetopic algebras

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We resort to simplicial sets as a model for the notion of homotopy. So we consider simplicial presheaves over $\mathbb{\Lambda}$:

$$\mathcal{S}\mathrm{p}(\mathbb{\Lambda}) := [\mathbb{\Lambda}^{\mathrm{op}}, \mathcal{P}\mathrm{sh}(\mathbb{\Delta})].$$

# Homotopy-coherent opetopic algebras

Consider $\mathcal{S}\mathrm{p}(\mathbb{A})_v$, the model structure on

$$\mathcal{S}\mathrm{p}(\mathbb{A}) = [\mathbb{A}^{\mathrm{op}}, \mathcal{P}\mathrm{sh}(\mathbb{\Delta})]$$

induced by the Reedy structure of $\mathbb{A}$ and the usual model structure on $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$.

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- A *Segal space* is a fibrant object in the left Bousfield localization

$$\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Segal}} = \mathcal{S}\mathrm{p}(\mathbb{\Lambda})_v[S^{-1}].$$

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- A *complete Segal space* is a fibrant object in the left Bousfield localization

$$\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Rezk}} = \mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Segal}}[\mathsf{R}^{-1}].$$

# Homotopy-coherent opetopic algebras

**Theorem**

The adjunction

$$(-)^{\mathrm{disc}} : \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})_\infty \underset{\perp}{\rightleftarrows} \mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Rezk}} : (-)_{-,0}$$

is a Quillen equivalence.

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The proof essentially generalizes the strategy of Joyal and Tierney in the simplicial case ($n = 1$) [Joyal and Tierney, 2007].

# Summary

To summarize:

- the “right” shape category for opetopic algebras

| | | | |
|---|---|---|---|
| Categories | $\mathcal{G}\mathrm{raph}$ | $\overset{\text{upgrade}}{\rightsquigarrow}$ | $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ |
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- $\infty$-opetopic algebras are the fibrant objects of $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})_\infty$, i.e. presheaves having the (non unique) lifting property against S;
- the idea of weak opetopic algebra is also modelled by *complete Segal spaces*:

$$(-)^{\mathrm{disc}} : \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})_\infty \underset{\perp}{\overset{\sim}{\rightleftarrows}} \mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Rezk}} : (-)_{-,0}.$$

Thank you for your attention!

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