2020-12-2 G0 Seminar

Polynomial Functors & Opetopes

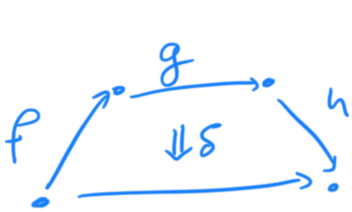
G0 Seminar
2/12/2020

1) Opetopes, informally

IDEA: Represent higher-dimensional composition laws

Example • Dimension 1 ≃ Category theory

γ ⇝ witness of the fact that gf is the sequential composition of g and f —

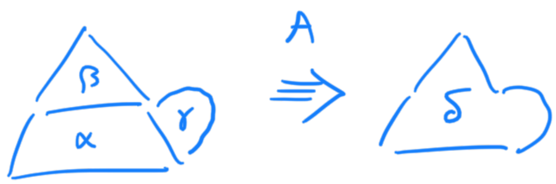


The "witness" approach is fundamentally unbiased —

hgf

Nullary composition correspond to the notion of identities

- Dimension 2 $\simeq$ Operads (a.k.a. multicategories)



This time, A witnesses the fact that

$\delta = \alpha(id, \beta, \gamma)$

Opetopes provide a geometrical formalism to talk about composition schemes in all dimensions.

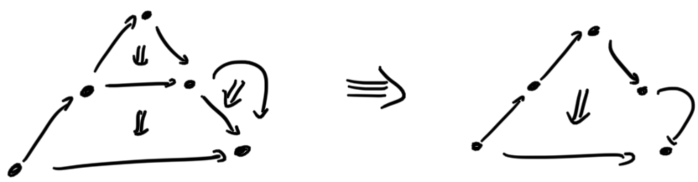
<u>Definition</u> (tentative)

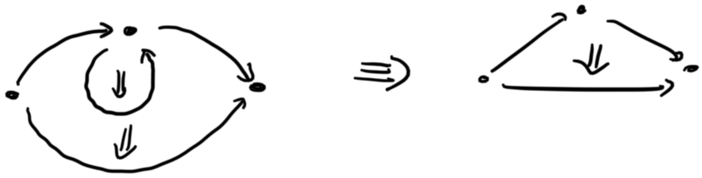
- There is a unique 0-dimensional opetope

- There is a unique 1-dimensional opetope

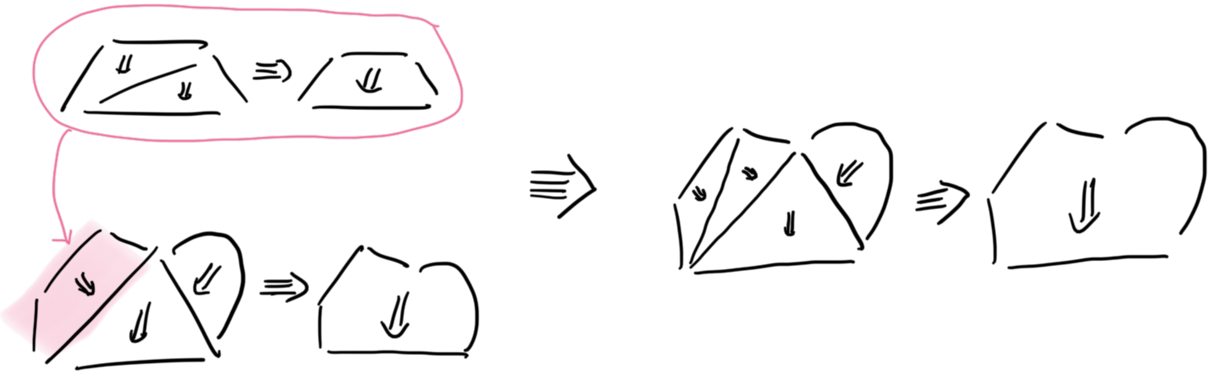
– A 2-dimensional opetope is essentially a pasting diagram of 1-opetopes

– A 3-dimensional opetope is essentially a pasting diagram of 2-opetopes





– A 4-dimensional opetope is essentially a pasting diagram of 3-opetopes

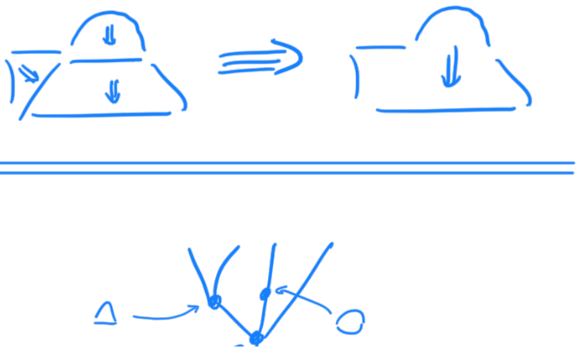


— etc ...

Problem The graphical approach is not formal and not even convenient –
How to define opetopes formally ?

2) Polynomial functors & trees

Motivation : Opetopes are fundamentally arborescent



↪ Need a good formalism for (decorated) trees!

Definition Polynomial (endo)functor

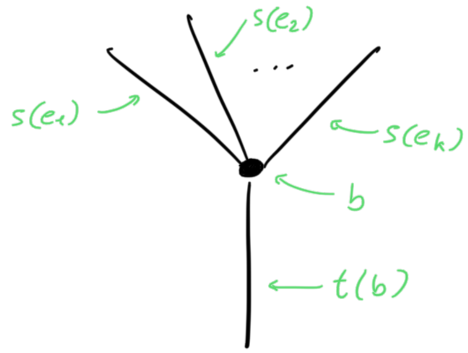
$$I \xleftarrow{s} E \xrightarrow{p} B \longrightarrow I$$

I: colors/types/sorts; E: inputs; B: operations

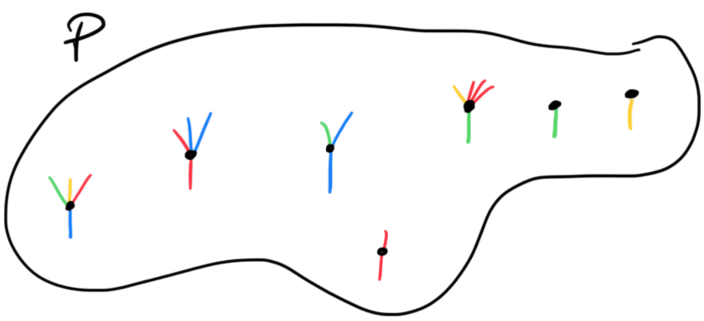
$b \in B$

$E(b) := p^{-1}(b)$
$= \{e_1, \ldots, e_k\}$

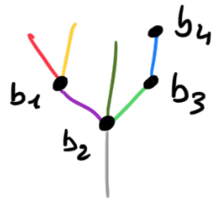
$\rightsquigarrow$



So a p.f. P is just a collection of operations symbols and types



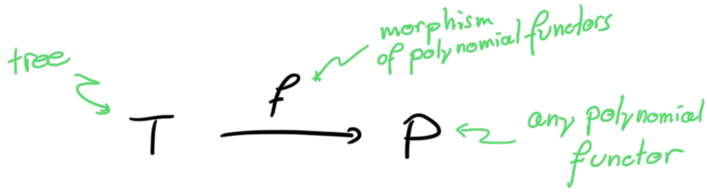
If the operations come together nicely, then visually, P forms a tree



Nicely = the colors describe binary adjacencies; connectivity; existence of a root; well founded; finite

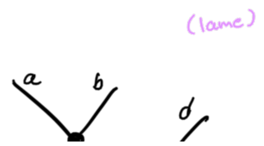
In this case, P is a <u>polynomial tree</u> —

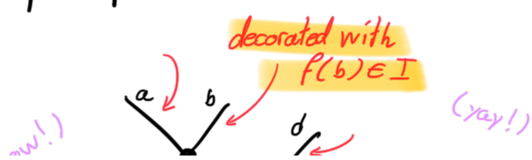
(important) <u>Definition</u>



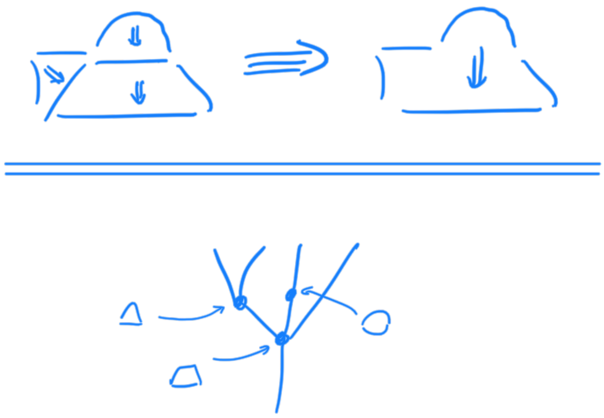
f decorates the nodes of T with operations of P, and edges of T with colors of P —

T $\qquad\qquad$ $T \xrightarrow{f} P$



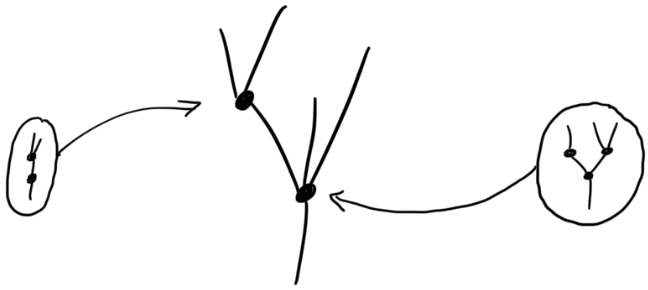


Let's come back to opetopes for a minute



Opetopes are trees decorated by opetopes!

Final step: Decorate trees with trees



3) The Baez-Dolan construction

Let P be a polynomial functor

$$P \quad = \quad I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I$$

Recall that if T is a P-tree, as in

$$T \xrightarrow{f} P$$

then

- the nodes of T are decorated by elements of B
- the edges ________________________ I

If we want trees of P-trees, we must consider Q-trees, where Q is of the form

$$Q \;=\; ? \xleftarrow{?} ? \xrightarrow{?} \mathrm{tr}P \xrightarrow{?} ?$$

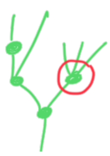
Definition If P is a polynomial monad

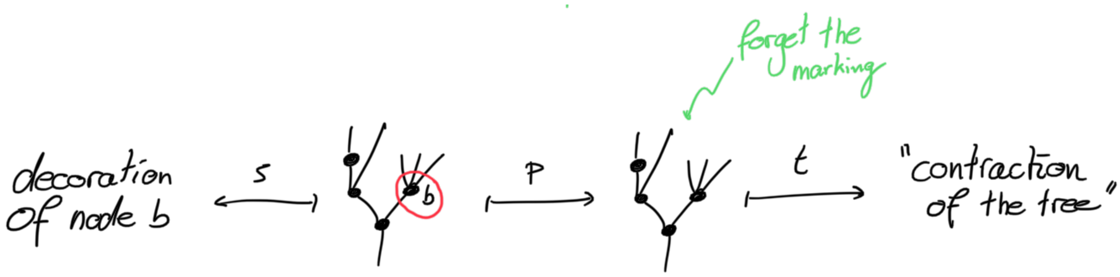
$$P \quad = \quad I \leftarrow E \longrightarrow B \longrightarrow I$$

then let

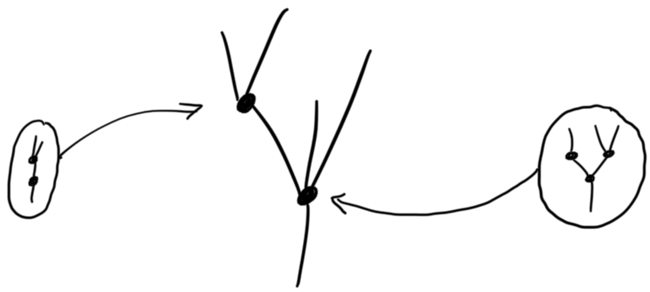
$$P^+ \quad = \quad B \xleftarrow{s} \mathrm{tr}^\bullet P \xrightarrow{p} \mathrm{tr}P \xrightarrow{t} B$$

P-trees with a marked node

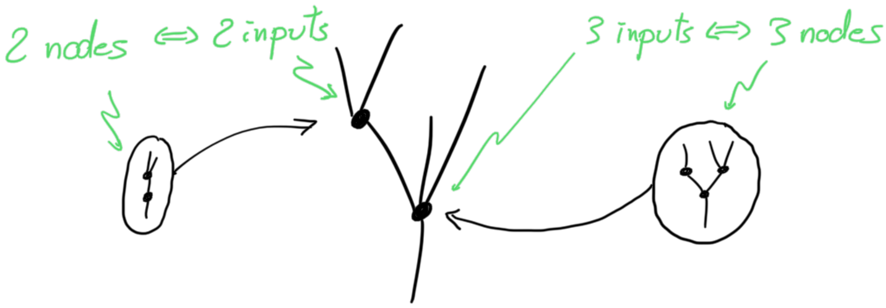


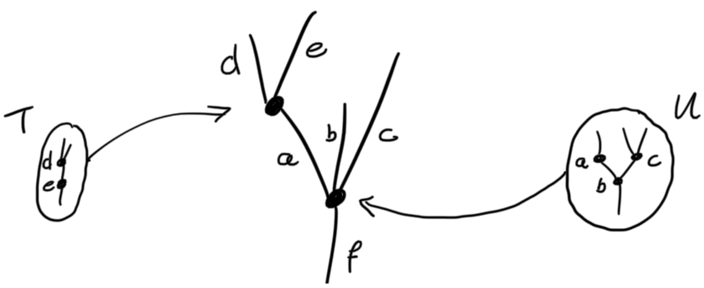


Consequences 1) P^+ - trees really are trees of P - trees



2) There are built-in "well-formedness" constraints





$a, b, \dots, f \in B$

$T, U \in \mathrm{tr}\, P$

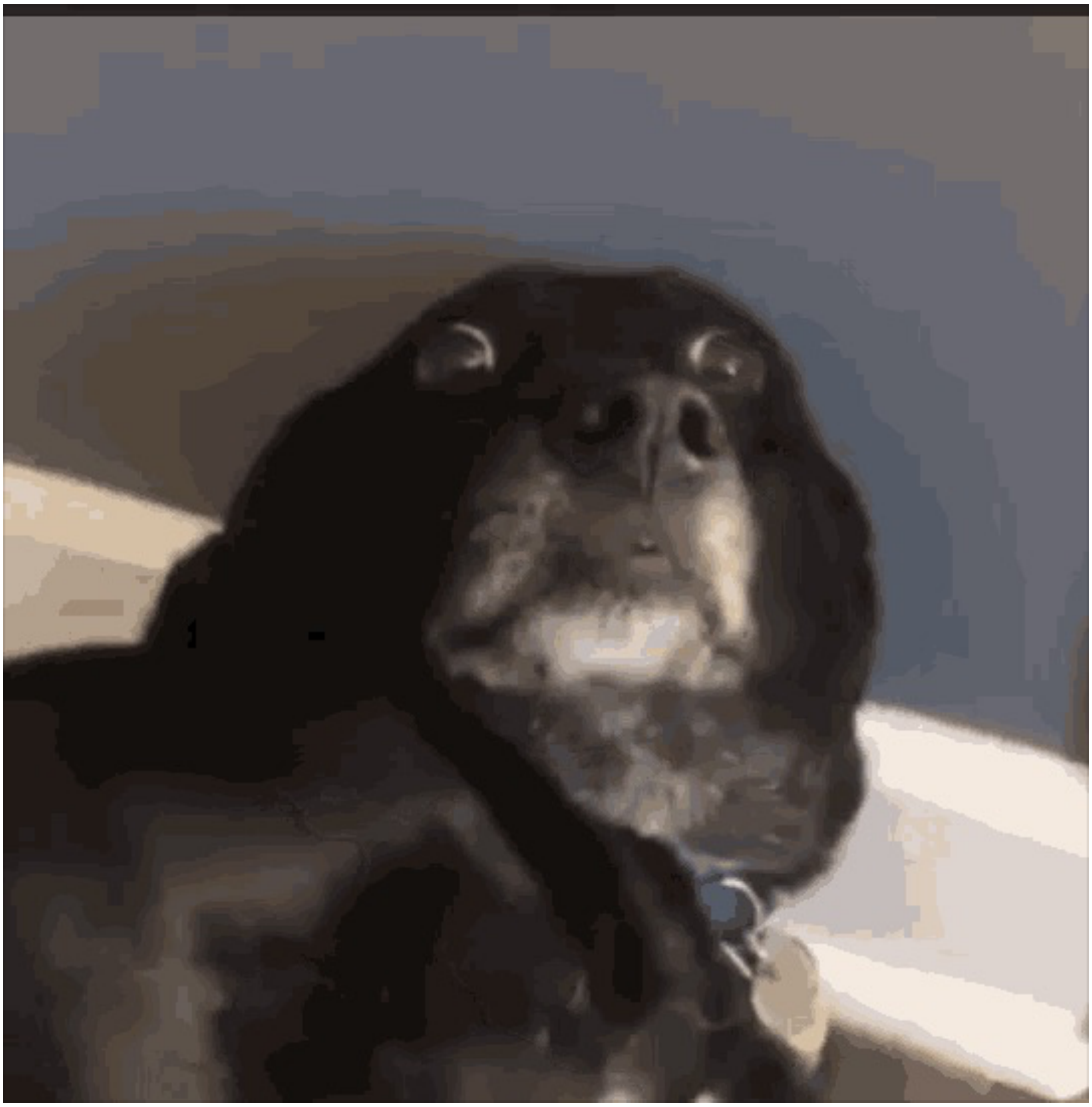
By construction,

$$\begin{cases} t(T) = a \\ t(U) = f \end{cases}$$

recall: t "contracts" P-trees

Theorem If P is a polynomial monad, then so is P^+

Corollary Here comes $P^{++++++++++++++\cdots}$



4) Polynomial functors & Opetopes

<u>Definition</u> Let $\mathfrak{Z}^0$ be the identity polynomial monad on <u>Set</u>

$$\mathfrak{Z}^0 = \{\blacklozenge\} \longleftarrow \{*\} \longrightarrow \{\bullet\} \longrightarrow \{\blacklozenge\}$$

Let $\mathfrak{Z}^{n+1} := (\mathfrak{Z}^n)^+$

An n-opetope is a color of $\mathfrak{Z}^n$

Consequences 1) If $n \geq 1$, an n-opetope is an operation of $\mathfrak{Z}^{n-1}$.

2) If $n \geq 2$, an n-opetope is a $\mathfrak{Z}^{n-2}$-tree.

3) We can write

$$\mathfrak{Z}^n = \mathbb{O}_n \xleftarrow{s} E_{n+1} \xrightarrow{p} \mathbb{O}_{n+1} \xrightarrow{t} \mathbb{O}_n$$

$\mathbb{O}_n$: set of n-opetopes

E_{n+1}: $(n+1)$-opetopes with a marked node

$\mathbb{O}_{n+1}$: set of $(n+1)$-opetopes

Let's unfold the definition.

n = 0

$$\mathfrak{Z}^0 = \{\blacklozenge\} \longleftarrow \{*\} \longrightarrow \{\bullet\} \longrightarrow \{\blacklozenge\}$$

There is a unique 0-opetope called the point and denoted by $\blacklozenge$

Geometrically •

n = 1

n = 1

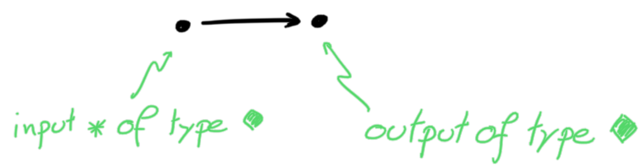
$$\mathfrak{Z}^0 = \{◆\} \longleftarrow \{*\} \longrightarrow \{■\} \longrightarrow \{◆\}$$

There is a unique 1-opetope called the arrow and denoted by ■

$\mathfrak{Z}^0$ says that ■ has one input $*$ of type ◆, and that its output type is ◆

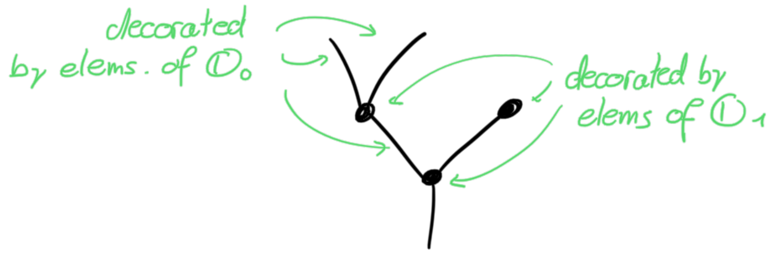
Geometrically,

■ =



n = 2

By definition, a 2-opetope is a $\mathfrak{Z}^0$-tree

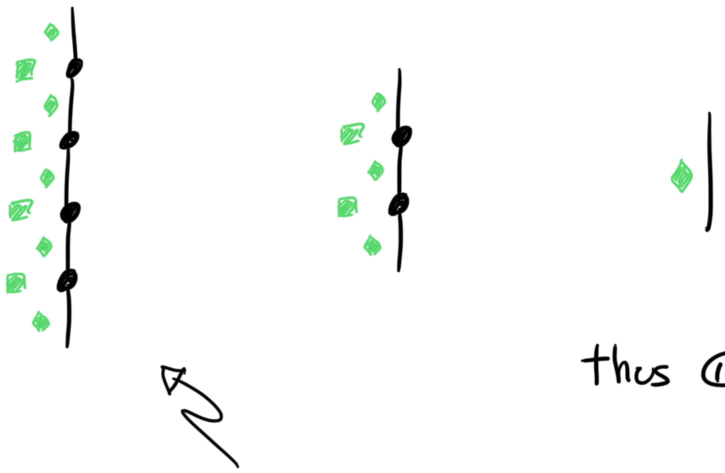


BUT

- ◆ is the only element of $\mathbb{O}_0$
- ■ is the only element of $\mathbb{O}_1$

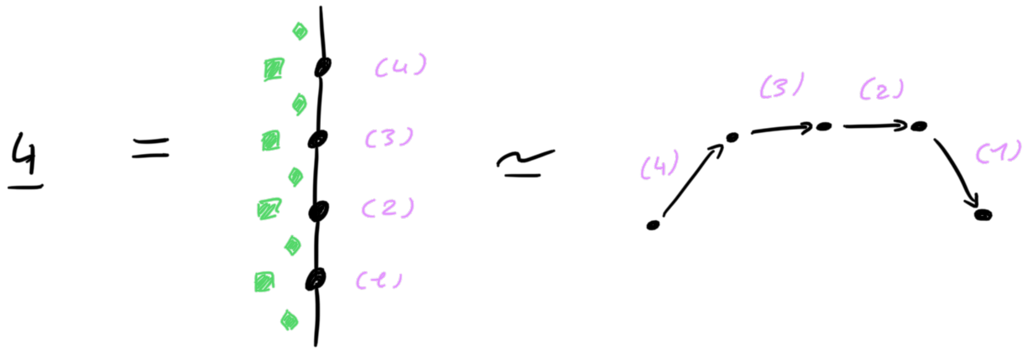
- ■ only has 1 input

So the 3°-trees necessarily look like



thus $\mathbb{O}_2 \cong \mathbb{N}$

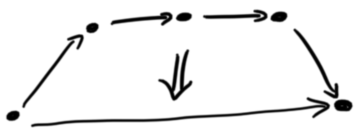
Geometrically, this tree talks about 4 arrows glued end to end



Arrows (1), (2), (3), (4) are now the inputs of the 2-opetope $\underline{4}$.

What is the output of $\underline{4}$?

By definition, $t(\underline{4}) \in \mathbb{O}_1$, so $t(\underline{4}) =$ ■ which we represent by

$\underline{4}$ = 

Likewise

$\underline{2}$ =  = 

$\underline{0}$ =  = 

Finally, $\mathfrak{Z}^1 = (\mathfrak{Z}^0)^+$ looks like this

$$\{\blacksquare\} \xleftarrow{s} E_2 \xrightarrow{p} \mathbb{N} \xrightarrow{t} \{\blacksquare\}$$

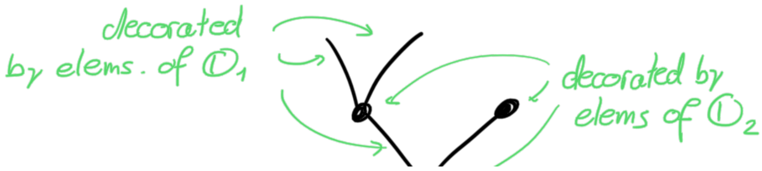
$\{\blacksquare\}$: $\mathbb{O}_1$

$\mathbb{N}$: $\{\underline{0}, \underline{1}, \underline{2}, \underline{3}, \dots\}$
$= \{$  $, \dots\}$

E_2: $p^{-1}(\underline{k}) = \{(1), (2), \dots, (k)\}$

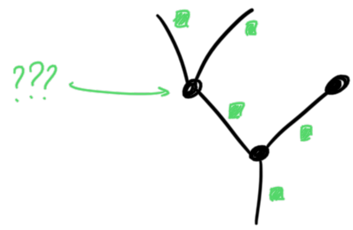
n=3

By definition, a 3-opetope is a $\mathfrak{Z}^1$-tree



so all edges are decorated by ■ **BUT** this time, we have more options for the nodes

Which 2-opetope could decorate this node?

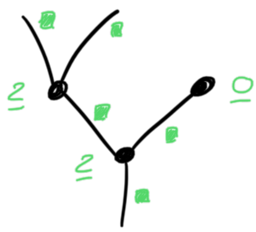


Since this node has 2 inputs, **so must the decorating opetope.** The only choice is

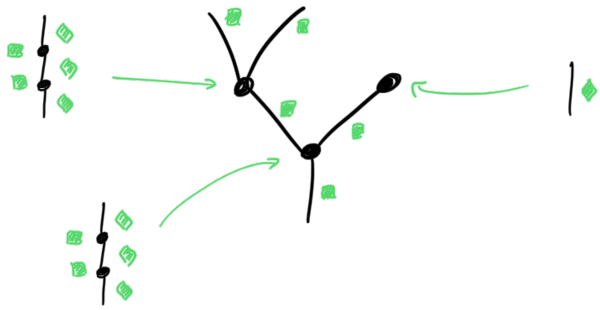
$\underline{2}$ =  = 

Likewise, the only 2-opetope that can decorate a node with 0 inputs is $\underline{0}$

Finally,  admits only one decoration



Or, as a tree of 3° - trees,



In fact, any tree can be made into a $\mathfrak{Z}^1$-tree in a unique way, thus $\mathbb{O}_3 \cong$ Trees

Finally, $\mathfrak{Z}^2 = (\mathfrak{Z}^1)^+ = (\mathfrak{Z}^0)^{++}$ looks like this

$$\mathbb{N} \xleftarrow{s} E_3 \xrightarrow{p} \text{Trees} \xrightarrow{t} \mathbb{N}$$

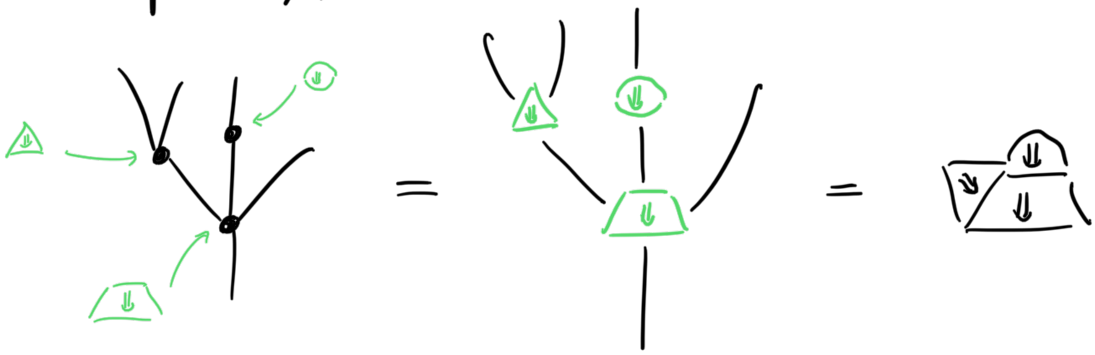
where for $T \in \mathbb{O}_3 = \text{Trees}$,

- $p^{-1}(T) =$ the set of nodes of T
- if $x \in p^{-1}(T)$, then

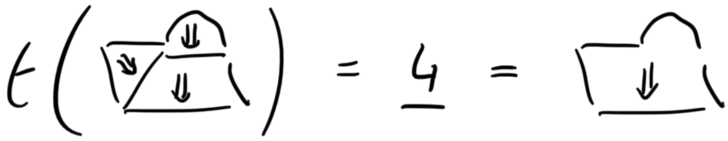
  $s(x) =$ number of inputs of x
- $t(T) =$ number of leaves of T

In $\mathfrak{Z}^2$, the nodes of T become its <u>inputs</u>

Graphically,

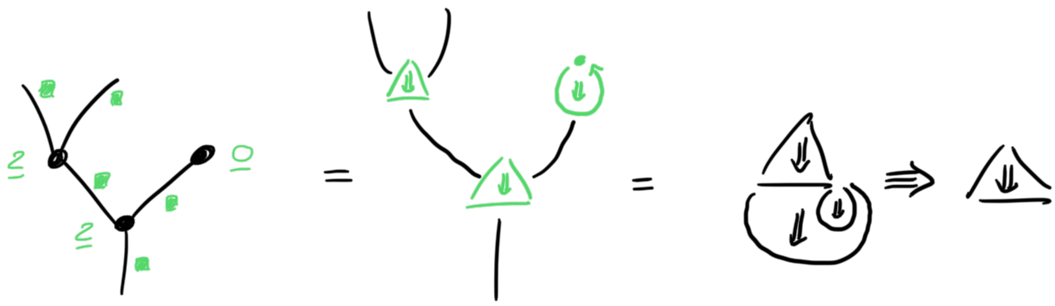


and since the tree has 4 leaves,



and finally we draw

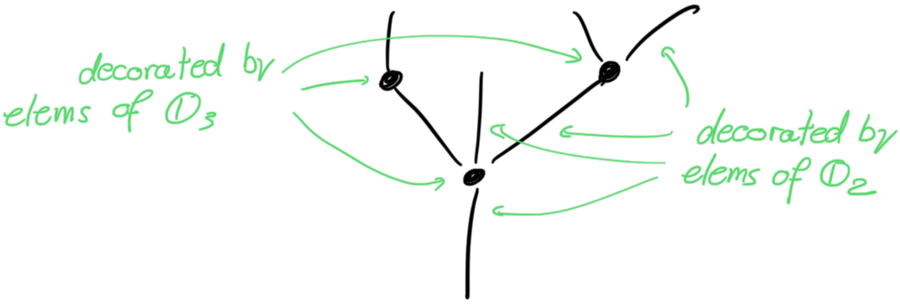
Other example



<u>$n = 4$</u>

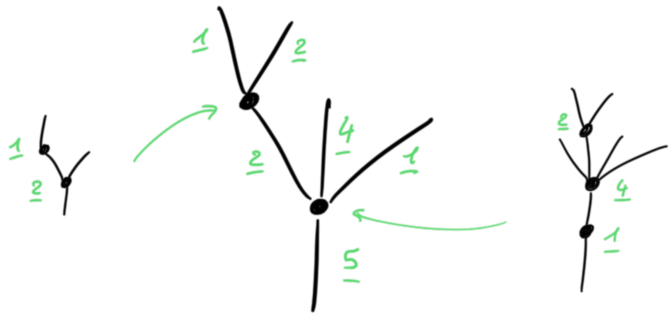
By definition a 4-[illegible] is a 3^2 tre

By definition, a 4-opetope is a 3-tree



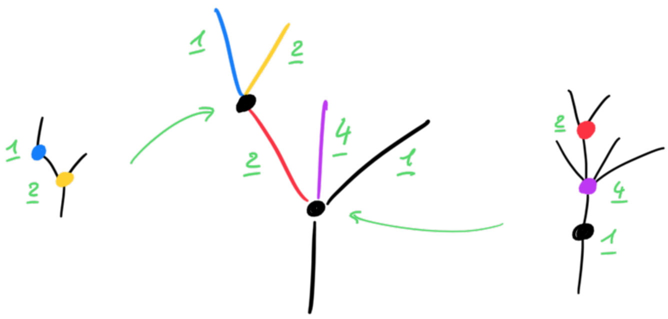
From now on the edge decorations are non-trivial

Small example

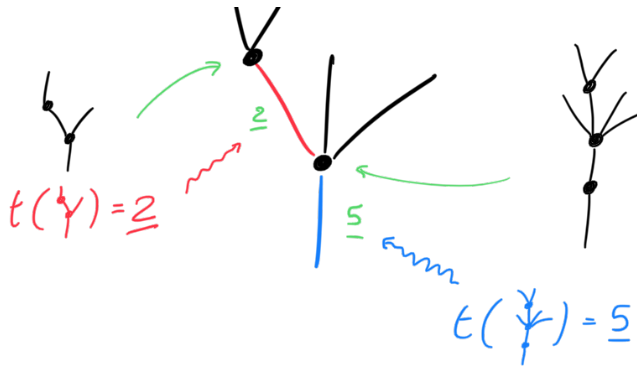


Things to note:

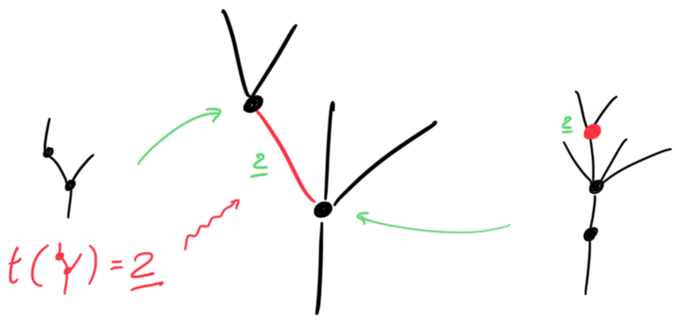
1) Decoration of input edges



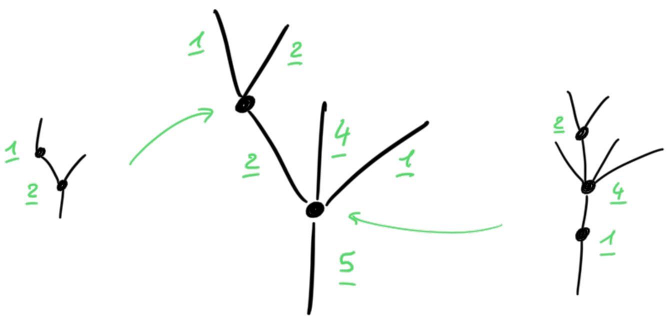
2) Decoration of output edges



3) Nodes <u>must</u> agree on the decoration of inner edges!



4) The main tree talks about gluing the $\mathbb{O}_3$ - opetopes decorating its nodes



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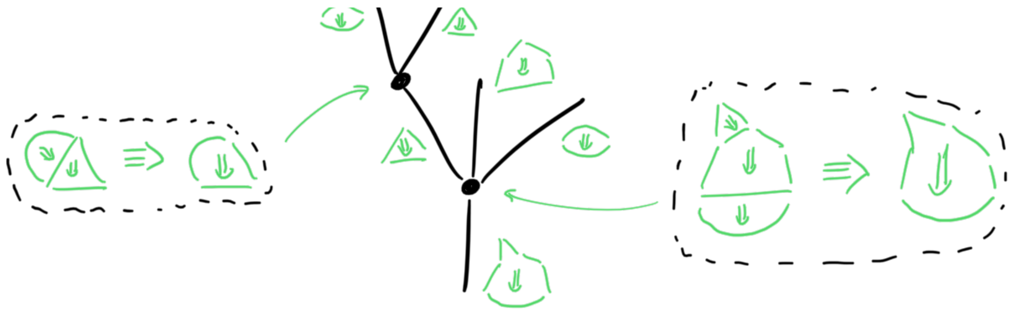

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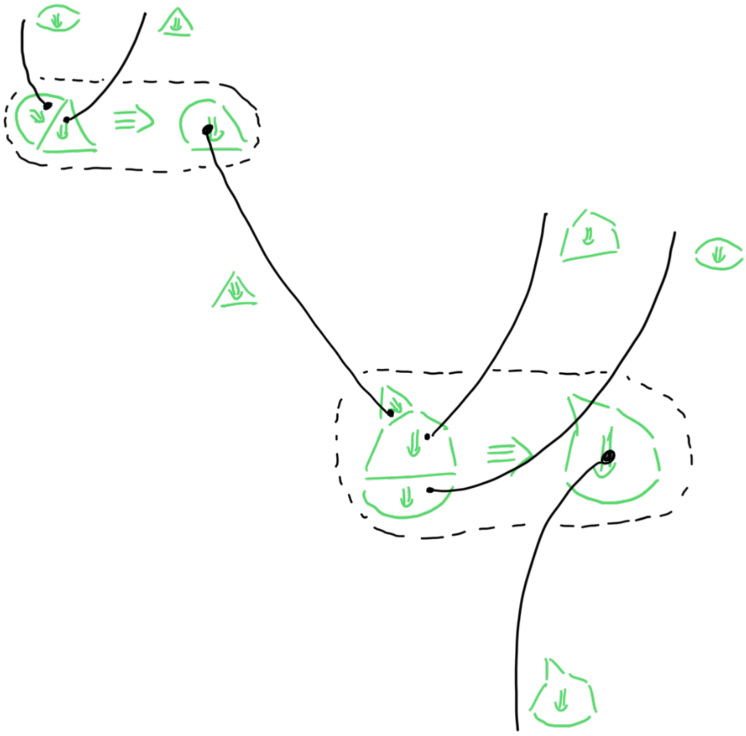

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