About Opetopes

Cédric Ho Thanh
National Institute for Informatics, Tokyo

1. Introduction

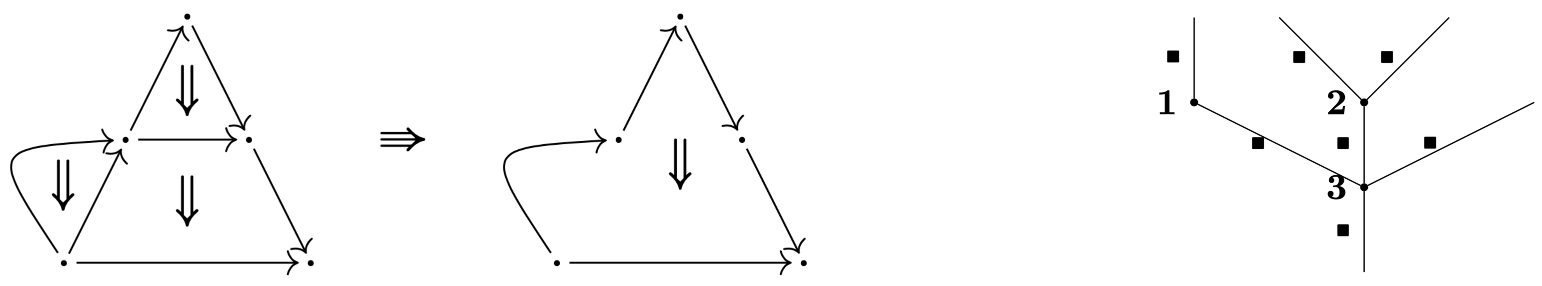
Opetopes are shapes (akin to globules, cubes, simplices, dendrices, etc.) introduced by Baez and Dolan to describe laws and coherence cells in weak ω-categories [1]. Their name reflects the fact that they encode the possible shapes for higher-dimensional operations:

opetope = **oper**ation poly**tope**

These shapes are attractive because they are easy to find "in nature", but they are difficult to manipulate efficiently. My thesis [2] is dedicated to lay the foundations of a more amenable theory of opetopes, first by carefully reviewing the approach of Kock et. al. [3], and by applying it along two main axes: syntax and algebra.

2. In a nutshell

Roughly speaking, opetopes are **trees of trees of trees of trees of...** While the classical image is something like on the left:

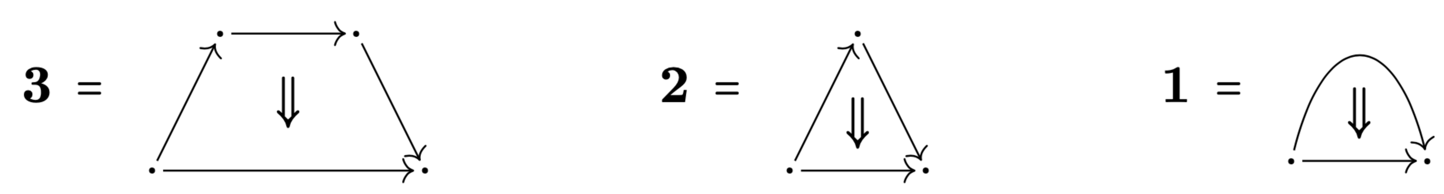


they are formally defined using a tree as on the right which is essentially the Poincaré dual of the geometrical shape above, but **decorated** with the tree representation of lower-dimensional faces [3]. This is thus an **inductive** definition, and the first few cases look like this.

- (Dimension 0 and 1) By definition, there are unique 0 and 1-dimensional opetopes, called the **point** and the **arrow**, respectively

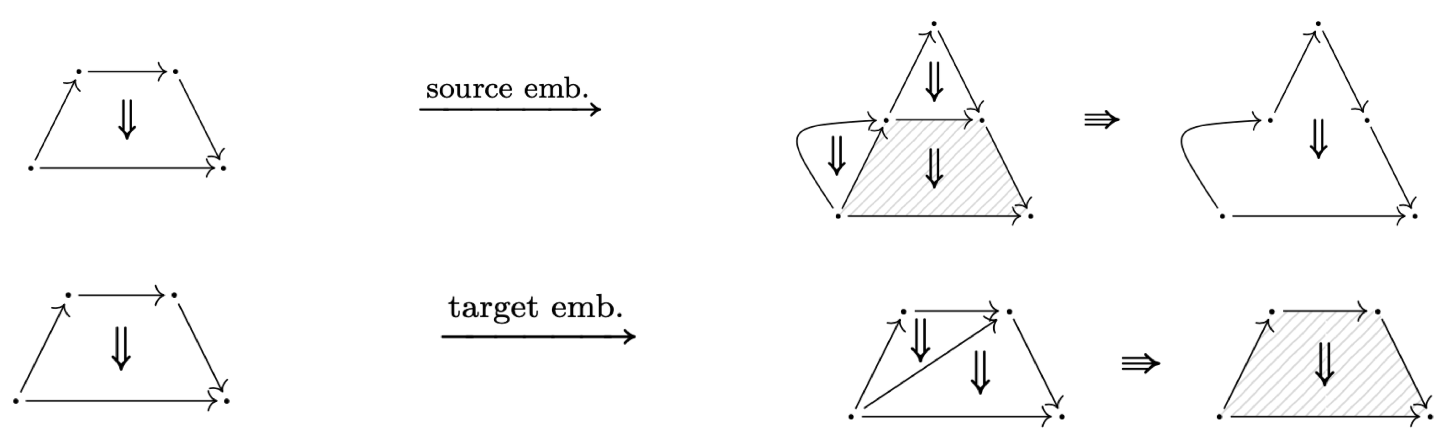
$$\cdot \qquad\qquad \cdot\longrightarrow\cdot$$

- (Dimension 2) Now, the induction starts. A 2-opetope is essentially a **well-formed pasting diagram** (or rather, a filler thereof) of 1-dimensional opetopes, i.e. a gluing of several instances of the arrow, glued end-to-end along the point. Examples include the following:



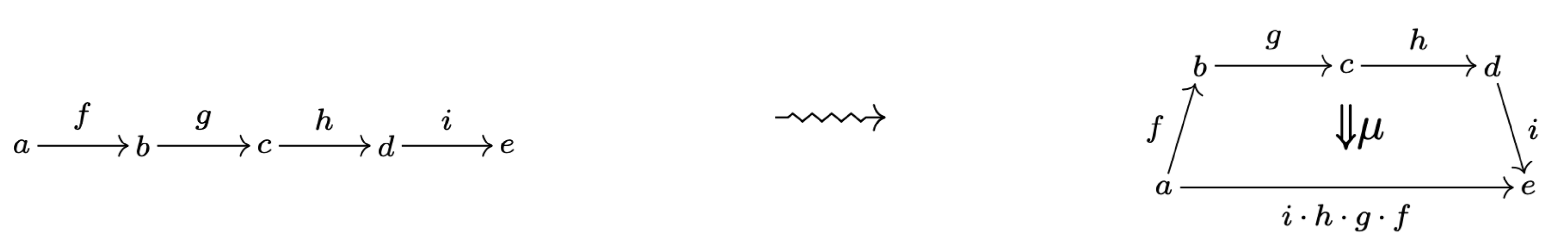
- (Dimension $n+1$) An $(n+1)$-opetope is simply a **pasting diagram** of n-opetope, or more formally, a **tree**, whose nodes are n-opetopes, and edges are $(n-1)$-opetopes.

There is a nice category $\mathbb{O}$ of opetopes [2, 4] which completely encodes the geometrical intuition behind them. Its morphisms are formal **source** and **target embeddings**:

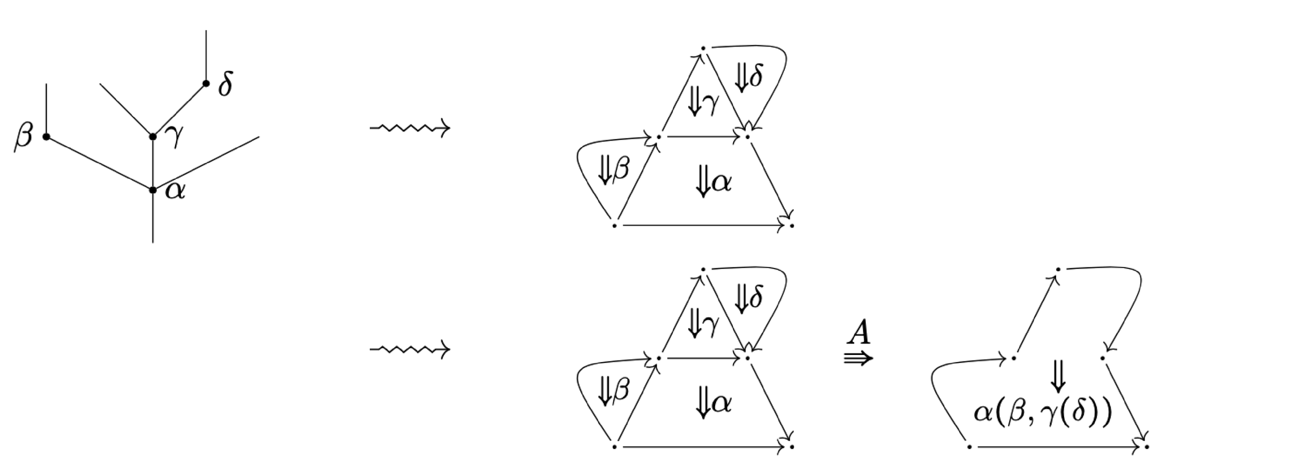


3. Algebras

A category is an algebraic structure where **operations** (a.k.a. morphisms) are shaped like arrows, in that their input and output consist of a single **color** (a.k.a. object). The action of composing those operations takes as input sequences of (composable) arrows, which can be seen as linear trees of operations. Thus, the **shapes of compositions** are linear trees, i.e. 2-opetopes.



One dimension above, we have (planar, colored) operads. Here, operations are shaped like 2-opetopes, since their inputs are sequences of colors. As before, composition takes as input a pasting diagram of operations, and since they are shaped like 2-opetopes, composition itself is shaped like a 3-opetope:



4. Algebras (cont.)

Continuing this pattern, in an n-dimensional **opetopic algebra**, the operations are shaped like n-opetopes, while compositions are shaped like $(n+1)$-opetopes.

Algebraic structure	Sets =0-algebras	Categories =1-algebras	Operads =2-algebras	$\mathbb{PT}$−combinads =3-algebras	…
Arity of compositions	Trivial =1-opetopes	Lists =2-opetopes	Trees =3-opetopes	Trees of trees =4-opetopes	…

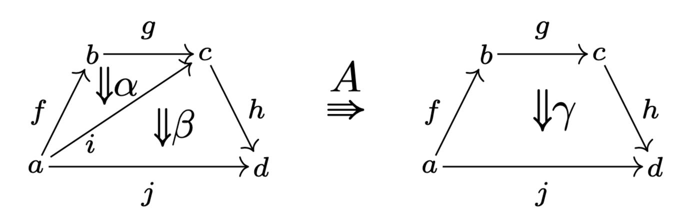
Theorem. *The category of n-opetopic algebra is a reflective subcategory of $\mathcal{P}\mathrm{sh}(\mathbb{O})$, the category of opetopic sets. In particular it is locally finitely presentable.*

Using the theory of monads with arities [5], we can upgrade $\mathbb{O}$ to a more complete shape category Λ_n.

Theorem. *There is a Cisinski model structure on $\mathcal{P}\mathrm{sh}(\Lambda_n)$, such that the fibrant objects correspond to the notion of "weak opetopic algebra". This structure generalizes the Joyal model structure on simplicial sets ($n=1$), and the Cisinski–Moerdijk structure on dendroidal sets ($n=2$).*

5. Syntax

Trees are data structures that are ubiquitous in computer science. Since opetopes are trees (of opetopes), we can hope for a representation amenable for computerized manipulations. The process is fairly simple: given an opetope, **name** all its faces



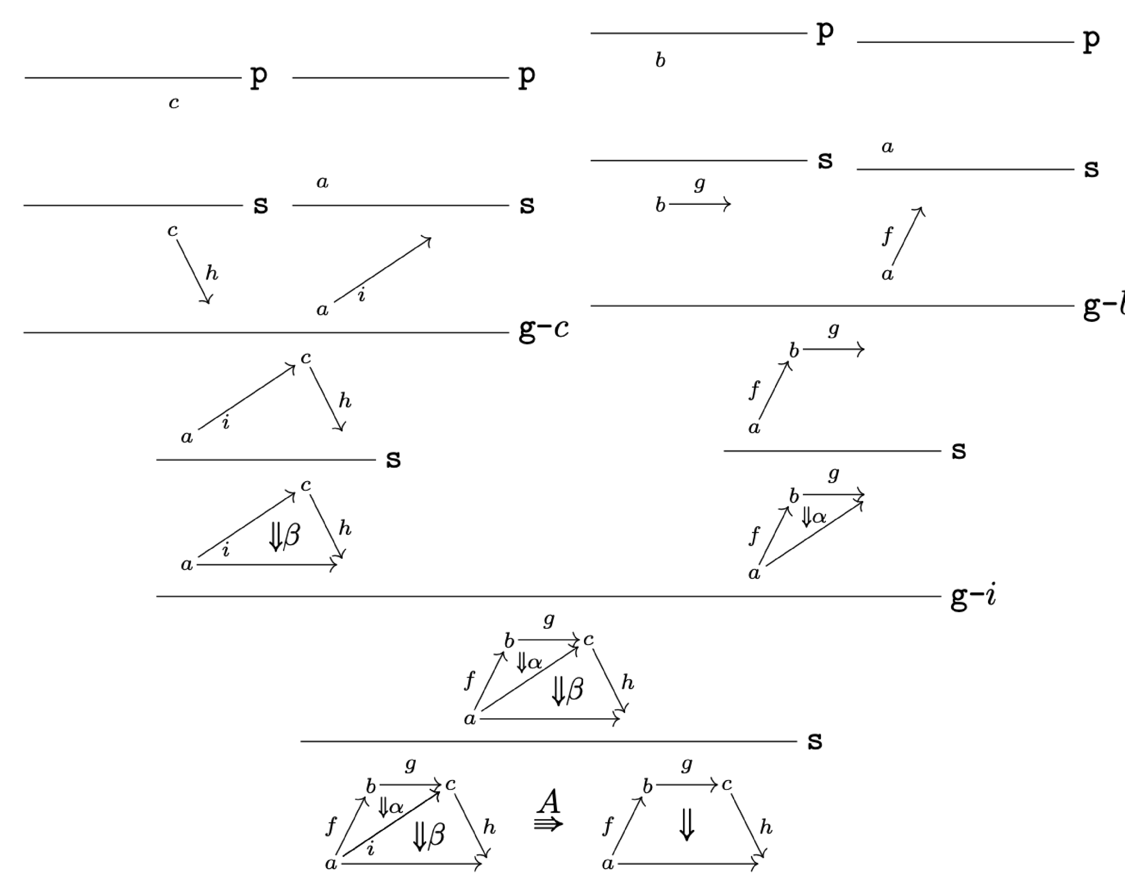
and write down a term that represents the adjacency relation among these faces:

$$A : \underbrace{\beta(i \leftarrow \alpha)}_{\text{source of } A} \multimap \underbrace{h(c \leftarrow g(b \leftarrow f))}_{\text{source of the source}} \multimap \underbrace{a}_{\text{etc.}} \multimap \underbrace{\varnothing}_{\text{end}} .$$

The question is: how to characterize the expressions that are geometrically meaningful? In [6], we present $\textsc{Opt}^!$, a sequent calculus that exactly solved this problem: the syntactic form of the opetope above is derived as

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash c:\varnothing}\,\mathtt{p}}{\vdash h : c \multimap \varnothing}\,\mathtt{s} \qquad \dfrac{\dfrac{}{\vdash a:\varnothing}\,\mathtt{p}}{\vdash i : a \multimap \varnothing}\,\mathtt{s}}{\vdash h(c\leftarrow i) : a \multimap \varnothing}\,\mathtt{g}\text{-}c}{\vdash \beta : h(c\leftarrow i) \multimap a \multimap \varnothing}\,\mathtt{s} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash b:\varnothing}\,\mathtt{p}}{\vdash g : b \multimap \varnothing}\,\mathtt{s} \qquad \dfrac{\dfrac{}{\vdash a:\varnothing}\,\mathtt{p}}{\vdash f : a \multimap \varnothing}\,\mathtt{s}}{\vdash g(b\leftarrow f) : a \multimap \varnothing}\,\mathtt{g}\text{-}b}{\vdash \alpha : g(b\leftarrow f) \multimap a \multimap \varnothing}\,\mathtt{s}}{\vdash \beta(i\leftarrow\alpha) : h(c\leftarrow g(b\leftarrow f)) \multimap a \multimap \varnothing}\,\mathtt{g}\text{-}i}{\vdash A : \beta(i\leftarrow\alpha) \multimap h(c\leftarrow g(b\leftarrow f)) \multimap a \multimap \varnothing}\,\mathtt{s}$$

(for simplicity, some details have been removed from the proof tree, and the rule names have been shortened). This corresponds to the following intuitive way of constructing it:



This system can be slightly modified to characterize finite opetopic sets as well. A prototype implementation in Python is available at `github.com/altaris/opetopy`.

6. References

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[2] C. HT, "Opetopes: syntactic and algebraic aspects" (University of Paris, Paris, France, Oct. 15, 2020), 341 pp.

[3] J. Kock, A. Joyal, M. Batanin, and J.-F. Mascari, "Polynomial functors and opetopes", Advances in Mathematics **224**, 2690–2737 (2010).

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[6] P.-L. Curien, C. HT, and S. Mimram, "A Sequent Calculus for Opetopes", in LICS 19: Proceedings of the 34th Annual ACM/IEEE Symposium on Logic in Computer Science (2019).