2021-04-14 G0 seminar

Tensor product of Lawvere theories

⚠ Work in progress

1) Lawvere theories

- For $n \in \mathbb{N}$ let $n = [n] = \{0, 1, \dots, n-1\}$

- Let $\mathcal{N}_0 \subseteq \underline{Set}$ be the full subcategory spanned by $0, 1, \dots$

↪ It's a skeleton of $\underline{FinSet}$

↪ It has finite sums

↪ All objects are copowers of 1

→ So $\mathcal{N}_0^{op}$ has the dual properties

A **Lawvere theory** $\mathcal{L}$ is a small category with a functor

$$\mathcal{N}_0^{op} \longrightarrow \mathcal{L}$$

that

- is the identity on objects
- preserves products

A **morphism** $F : \mathcal{L} \longrightarrow \mathcal{K}$ is a functor s.t.

$$\begin{tikzcd} & \mathcal{N}_0^{op} \arrow[dl] \arrow[dr] & \\ \mathcal{L} \arrow[rr, "F"] & & \mathcal{K} \end{tikzcd}$$

If $\mathcal{C}$ is a category with finite products, then a $\mathcal{L}$-model in $\mathcal{C}$ is a product preserving functor

$$X : \mathcal{L} \longrightarrow \mathcal{C}$$

We write $X = X^1 := X1$, $X^n := Xn$

A morphism of models $f : X \longrightarrow Y$ is simply a natural transformation

$\rightsquigarrow$ $\mathcal{L}(\mathcal{C})$ category of $\mathcal{L}$-models in $\mathcal{C}$

$\hookrightarrow$ It also has finite products!

<u>Examples</u> The theory of groups

$$Grp = \mathcal{N}_0^{op}$$

(generators) $+ \begin{cases} e : 0 \longrightarrow 1 \\ m : 2 \longrightarrow 1 \\ i : 1 \longrightarrow 1 \end{cases}$

(relations) $+ \begin{cases} m(m \times id_1) = m(id_1 \times m) \\ m(e \times id_1) = id_1 = m(id_1 \times e) \\ m\langle i, id_1 \rangle = id_1 = m\langle id_1, i \rangle \end{cases}$

Models:

- $\mathrm{grp}(\underline{\mathrm{Set}})$ = Groups
- $\mathrm{grp}(\underline{\mathrm{Top}})$ = Topological groups
- $\mathrm{grp}(\underline{\mathrm{DiffMan}})$ = Lie groups
- $\mathrm{grp}(\mathrm{grp}(\underline{\mathrm{Set}}))$ = Abelian groups ?!

<u>Proposition</u> A morpism $\alpha : X \longrightarrow Y$ in $\mathcal{S}(\mathcal{C})$ is simply a morphism $\alpha : X \longrightarrow Y$ in $\mathcal{C}$ that is compatible with the generators –

for $f : m \longrightarrow n$ a generator

$$\begin{tikzcd} X^m \arrow[r, "f^m"] \arrow[d, "\alpha^m"'] & X^n \arrow[d, "\alpha^n"] \\ Y^m \arrow[r, "f"] & Y^n \end{tikzcd}$$

2) Distributivity

⚠ Not the same as distributive laws

For $m, n \geq 0$, let the **suffle permutation**

$$\tau_{n;m} \in S_{nm}$$

be determined by

$$\tau_{n;m}\Big(\underbrace{(j-1)n}_{(j-1)^{th}\text{ packet}} + \underbrace{i}_{\text{offset}}\Big) = \underbrace{(i-1)m}_{(i-1)^{th}\text{ packet}} + \underbrace{j}_{\text{offset}}$$

$\Rightarrow$ $\tau_{n;m}$ rearranges m packets on n elements into n packets on m elements

Example $m = 2$, $n = 3$

$j = 1$, $i = 3$

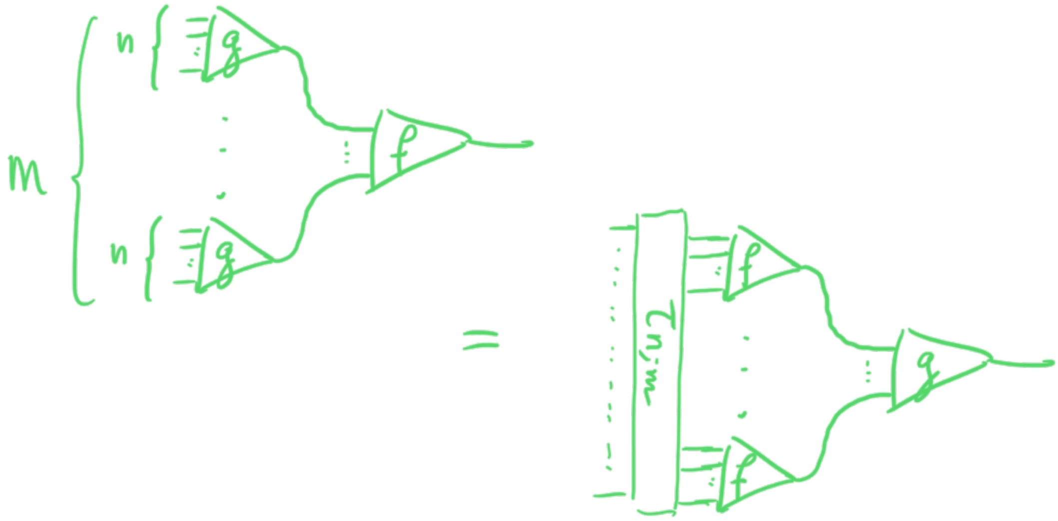
$(j-1)n + i = 3$ $\rightarrow$

$\leftarrow (i-1)m + j = 5$

If $X \in \mathscr{C}$, let $\tau_{X;n;m} : X^{mn} \longrightarrow X^{nm}$ be the suffle permutation applied to X.

Let $\mathcal{L}$ be a Lawvere theory, and $f : m \to 1$, $g : n \to 1$. We say that **f distributes over g**, written $f \boxtimes g$, if

$$f g^m = g f^n \tau_{n;m}$$



3) Tensor product

Let $\mathcal{L}$ and $\mathcal{K}$ be Lawvere theories. Their **tensor product $\mathcal{L} \otimes \mathcal{K}$** is given by

$$\mathcal{L} \otimes \mathcal{K} = \mathcal{N}_0^{op}$$

$+$ generators of $\mathcal{L}$ and $\mathcal{K}$

$+$ relations of $\mathcal{L}$ and $\mathcal{K}$

$+$ $f \boxtimes g$ for all $f \in \mathcal{K}(m,1)$ and $g \in \mathcal{L}(n,1)$

<u>Theorem</u> There is an equivalence
$$\mathcal{L} \otimes \mathcal{K}(\mathcal{C}) \simeq \mathcal{L}(\mathcal{K}(\mathcal{C}))$$

$\hookrightarrow$ Main argument: $f \boxtimes g$ for all $g \in \mathcal{K}/1$ encodes the fact that f is always a morphism of $\mathcal{K}$-models

<u>Corollary</u> $\mathcal{L}(\mathcal{K}(\mathcal{C})) \simeq \mathcal{K}(\mathcal{L}(\mathcal{C}))$

<u>Proof</u> Since $\tau_{n;m}^{-1} = \tau_{m;n}$, $f \boxtimes g \Leftrightarrow g \boxtimes f$

Thus $\mathcal{L} \otimes \mathcal{K} = \mathcal{K} \otimes \mathcal{L}$ $\blacksquare$

<u>Examples</u> - $\mathcal{G}rp \otimes \mathcal{G}rp = \mathcal{A}b$

- $\mathcal{A}b \otimes \mathcal{A}b = \mathcal{A}b$

- $Grp \otimes Mon \neq Ring$

since zero $= z \boxtimes u =$ unit reads

$$z\, u^{o} = u\, z^{o}\, \tau_{o;o}$$

i.e. $z = u$!

Other way to see it: if X is a group-objec in $\underline{Mon}$:

$$\begin{tikzcd}
X^2 \arrow[d, "m"] \\
X \arrow[loop right, "i"] \\
\{0\} \arrow[u, "u"']
\end{tikzcd}$$

$\{0\} \rightsquigarrow$ Trivial monoid

then since u is a morphism of monoids, it maps 0 to $0_X \rightsquigarrow$ multiplicative unit is forced to be 0_X :(

Back to $Grp \otimes Grp = Ab$

<u>Lemma</u> (Eckmann - Hilton argument)

Given a set X, two binary operations $\circ, \bullet : X^2 \longrightarrow X$, and $e \in X$ be a unit to both. If

$$(a \circ b) \bullet (c \circ d) = (a \bullet c) \circ (b \bullet d)$$

Then $\circ = \bullet$

↪ In fact, they are even associative and commutative

<u>Lemma</u> (EH, v.1.1)

Given a set X, two binary operations $m, m' : X^2 \longrightarrow X$, and $e \in X$ be a unit to both. If

$$m \boxtimes m'$$

Then $m = m'$

<u>Corollary</u> (EH, v.1.2) We have

$$\mathcal{M}on \otimes \mathcal{M}on = \mathcal{CM}on$$

Theory of monoids

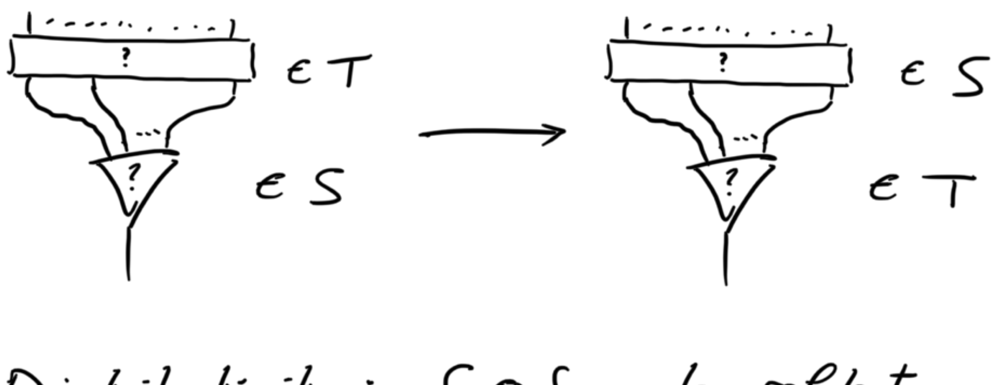
Theory of commutative monoids

<u>Proof</u> For e and e' the units of the left and right instance of $\mathcal{M}on$, the fact that $e \boxtimes e'$ means that $e = e'$.
From there, EH v.1.1 applies ▨

<u>Lemma</u> (EH v. 2.0) EH also works for n-ary operations.

4) Distributivity vs. distributive laws

- For T, U two monads, a distributive law is a way to "flip" operations of T and U:



- Distributivity in $\mathcal{L} \otimes \mathcal{S}x$ also reflects a notion of flipping, but much more constrained

5) Stability

We say that $\mathcal{L}$ is stable

- syntactically, if $\mathcal{L}^{\otimes k} \xrightarrow{\simeq} \mathcal{L}^{\otimes k+1}$
- semantically, if $\underbrace{\mathcal{L}^{\otimes k}(\underline{Set}) \xrightarrow{\simeq} \mathcal{L}^{\otimes k+1}(\underline{Set})}$

We also say that $\mathcal{L}^{\otimes k}$ and $\mathcal{L}^{\otimes k+1}$ are Morita equivalent

Remark Syntactical stability $\Rightarrow$ Semantic stability

Example

Example

- By EH, $\mathcal{M}on$ is syntactically stable with $k=2$:
  - $\mathcal{M}on \otimes \mathcal{M}on = \mathcal{CM}on$
  - $\mathcal{CM}on \otimes \mathcal{M}on = \mathcal{CM}on$

- Likewise, $\mathcal{G}rp$ is also syntactically stable with $k=2$

- <u>Corollary</u> $\mathcal{CM}on$ and $\mathcal{A}b$ are syntactically stable at $k=1$

- $\mathcal{M}ag_n$ is unstable (for $n \geq 1$)

Unique generator of arity n, and no relation

- $\mathcal{SG}rp$ is unstable

Semigroups, i.e. monoids without unit

これから先

建設中

です ^^

6) Attempt towards measuring syntactic stability

Let $\mathcal{L}$ be a Lawvere theory, and consider

$$\underbrace{\mathcal{A}b}_{} \otimes \mathcal{L}$$

Write 0 for the unit,
λ_n for the n-ary multiplication
and i for the inverse map

For $f : m \longrightarrow 1$ a morphism in $\mathcal{L}$
and $1 \leq i \leq m$, its i-th axis is

$$f^{(i)} := f(0^{i-1} \times id_1 \times 0^{n-i}) : 1 \longrightarrow 1$$

[BD'79]

Proposition (Boardman - Vogt decomposition)

In $\mathcal{A}b \otimes \mathcal{L}$, for $f : m \to 1$,

Can be weakened to $\mathcal{M}on$

$$f = \lambda_m \prod_{i=1}^{m} f^{(i)}$$

and the $f^{(i)}$ are unique for this property.

So $\mathcal{A}b \otimes \mathcal{L}$ is completely determined by

$$\varepsilon_1(\mathcal{L}) := \mathcal{A}b \otimes \mathcal{L}(1,1)$$

Importing the structure of $\mathcal{A}b$, $\varepsilon_1(\mathcal{L})$

is naturally a ring:

- $a + b = \lambda_2 \langle a, b \rangle$
- 0 is the unit of $\mathcal{A}b$
- $-a = i \cdot a$ $= a \cdot i$ since $a \boxtimes i$!
- $a \cdot b$ is the composition of a and b
- $1 = id_1$

<u>Proposition</u> $\mathcal{E}_1(\mathcal{L} \otimes \mathcal{S}_k) \cong \mathcal{E}_1(\mathcal{L}) \otimes_{\mathbb{Z}} \mathcal{E}_1(\mathcal{S}_k)$

<u>Proof</u> Follows from the fact that $\mathcal{A}b = \mathcal{A}b \otimes \mathcal{A}b$ ∎

<u>Proposition</u> If $\mathcal{L}$ is syntactically stable at k, then so is $\mathcal{E}_1(\mathcal{L})$

↳ i.e. $\mathcal{E}_1(\mathcal{L})^{\otimes_{\mathbb{Z}} k} \cong \mathcal{E}_1(\mathcal{L})^{\otimes_{\mathbb{Z}} (k+1)}$

<u>Examples</u>

$\mathcal{L}$	$\mathcal{E}_1(\mathcal{L})$	Can conclu is unstable
$\mathcal{M}ag_0$ (*)	$\mathbb{Z}$	No
$\mathcal{M}ag_n,\ n \geq 1$	$\mathbb{Z}\langle x_1, \dots, x_n \rangle$	Yes

$\mathcal{S}grp$	$\mathbb{Z}[x,y]/(x^2-x,\,y^2-y)$	Yes
$c\mathcal{S}grp$ (commutative semigroups)	$\mathbb{Z}[x]/(x^2-x)$	Yes
$\left.\begin{matrix}\mathcal{M}on,\ c\mathcal{M}on \\ \mathcal{G}rp,\ \mathcal{A}b\end{matrix}\right\}$	$\mathbb{Z}$	No
$\mathcal{M}od_R$, $\mathcal{A}lg_R$ (R a ring)	R	Depends on R

(*) But it is easy to see that $\mathcal{M}ag_0$ <u>is</u> syntactically stable at $k=1$.

7) Attempts at measuring semantic stability

We say that two Lawvere theories $\mathcal{L}$ and $\mathcal{M}$ are **Morita-equivalent** if

$$\mathcal{L}(\underline{Set}) \simeq \mathcal{M}(\underline{Set})$$

Necessary and sufficient conditions are already knows (similar to Morita-equivalence for rings), but not very tractable ...

For $\mathcal{L}$ a Lawvere theory, its **center $\mathcal{Z}(\mathcal{L})$** is the largest subtheory s.t.

$$\mathcal{Z}(\mathcal{L})/1 \boxtimes \mathcal{L}/1$$

$\hookrightarrow$ i.e. $\forall f \in \mathcal{Z}(\mathcal{L})(m,1)$, $g \in \mathcal{L}(n,1)$

$$f \boxtimes g$$

$\mathcal{L}$ is **commutative** if $\mathcal{Z}(\mathcal{L}) = \mathcal{L}$

Proposition If $\mathcal{L}$ is syntactically stable (at 1), then it is commutative.

Theorem from classical homological algebra

If R is a ring, then

$$Z(R) \cong Z(\mathcal{M}od_R)$$

↳ endomorphisms of $id_{\mathcal{M}od_R}$

Generalization Let $\mathcal{T}$ be the full subcategory of $[\mathcal{L}(\underline{Set}), \mathcal{L}(\underline{Set})]$ spanned by powers of $id_{\mathcal{L}(\underline{Set})}$. It is a Lawvere theory, and

$$Z(\mathcal{L}) \cong \mathcal{T}$$

Corollary If $\mathcal{L}$ and $\mathcal{M}$ are commutative and Morita-equivalent, then $\mathcal{L} \simeq \mathcal{M}$

Proof

$$\mathcal{L} = Z(\mathcal{L}) \cong \mathcal{T}_{\mathcal{L}} \simeq \mathcal{T}_{\mathcal{M}} \cong \mathcal{M}$$

$\blacksquare$

Corollary If $\mathcal{L}$ is commutative and semantically stable at some k, then it is syntactically stable at k –