Recurrence theorems for topological Markov chains

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iTHEMS, April 22nd 2022

Finite Makov chains

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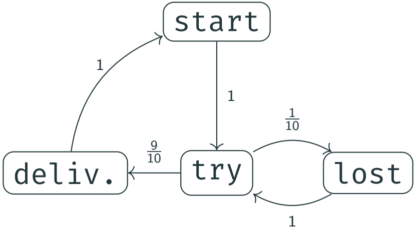

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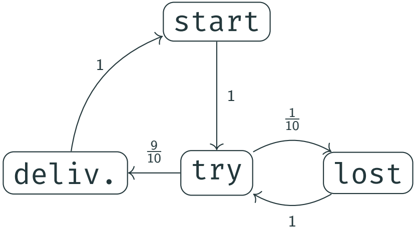

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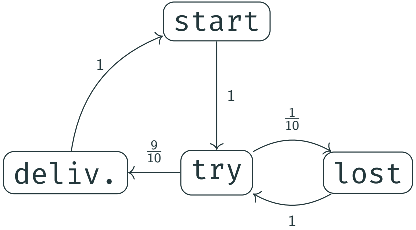

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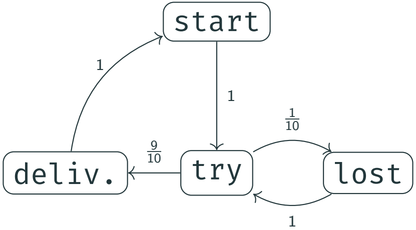

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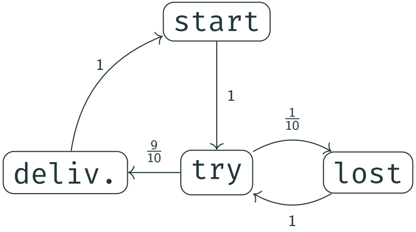

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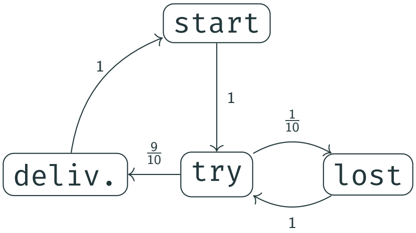
Sure reachability



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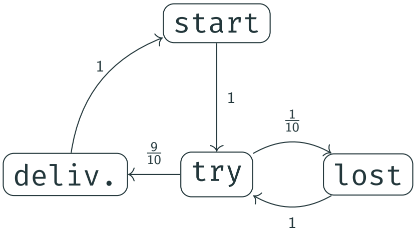

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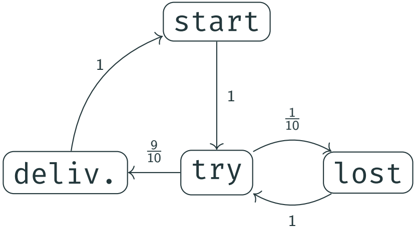

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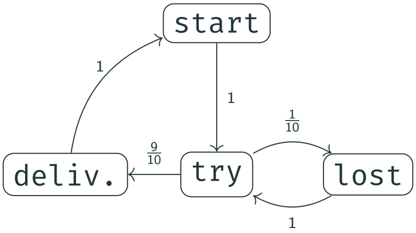

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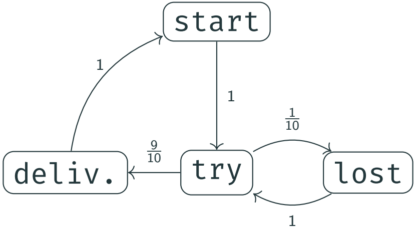

It seems obvious that starting from `start` (or indeed, any state), the desirable state `delivered` will almost surely be reached.

1. The probability to get from `start` to `delivered` is > 0;
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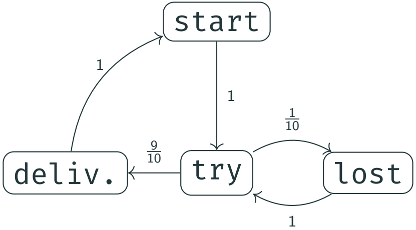
We can even say more: **every state is reachable from every other**.

Sure reachability

A chain where every state is reachable from every other is called **strongly connected**.



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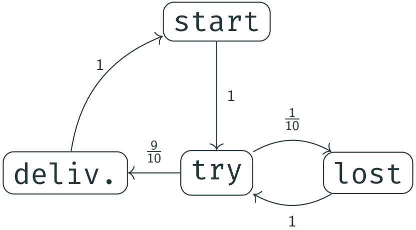
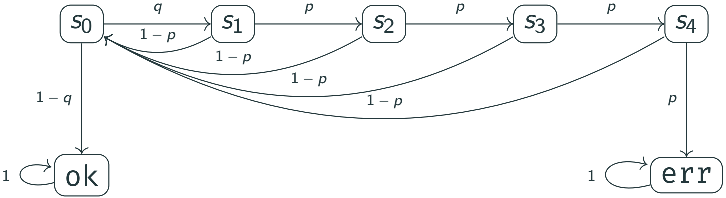




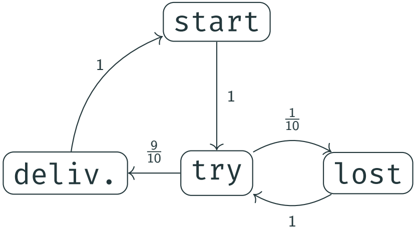
Sure reachability

Theorem (sure reachability)

If (X, γ) is a strongly connected finite Markov chain and $x \in X$ is a state, then the probability of eventually reaching x (starting from anywhere) is 1:

$$X \vDash \mathbb{P}(\lozenge x) = 1.$$

Recurrence



But that's not all. Not only do we almost surely reach `delivered`, but we almost surely reach it **infinitely often**.

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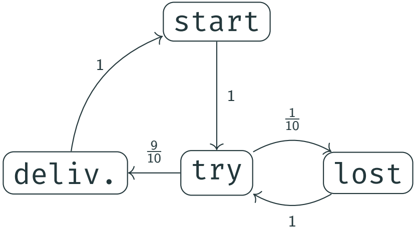
Recurrence

Key point

If a process evolves in a **finite** and **strongly connected** Markov chain, and A is a set of "good states", then the process is garanteed (in a probabilistic sense) to reach A infinitely often.



In our previous example, $A = \{\texttt{delivered}\}$.

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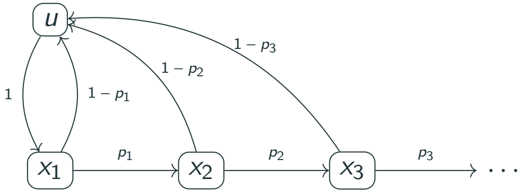
What if our probabilistic process has infinitely many states?



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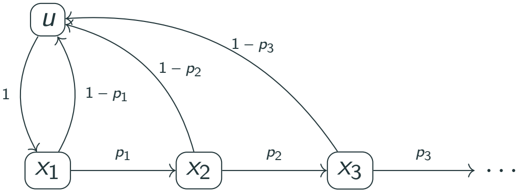

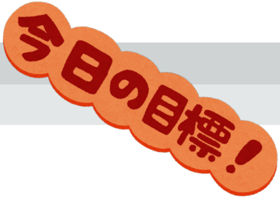

Today's objective

Generalize the recurrence theorem to infinite Markov chains.

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(usually, X is a discrete measurable space, so that every singleton $\{x\}$ is a measurable event)

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The generalization seems obvious:

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Example

$X = \mathbb{R}$, $\gamma(x) = \mathcal{N}(x, 1)$.

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To state a recurrence theorem, we also need a notion of **reachability**:

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But way a minute, in our previous example where $X = \mathbb{R}$ and $\gamma(x) = \mathcal{N}(x, 1)$, we have $\gamma(x, y) = 0$ for all $x, y \in X$. **No state is reachable from x!** (except x itself)

Infinite Markov chains (the right way)

Solution

Instead of focusing on wether or not $\gamma(x, y) > 0$, we should instead ask if $\gamma(x, U) > 0$ for any "arbitrary small set" $U \ni y$.

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Solution

Instead of focusing on wether or not $\gamma(x, y) > 0$, we should instead ask if $\gamma(x, U) > 0$ for any ~~"arbitrary small set"~~ **open set** $U \ni y$.

Of course we can't just take U to be a measurable set since in most cases $\{y\}$ is measurable... So we turn to **topology**.

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Nugget of wisdom 1

Topology + probability theory = Polish spaces

A Polish space is a topological space that is separable (it admits a dense countable subset) and completely metrizable (its topology is generated by a metric under which every Cauchy sequence converges).

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Here, Δ is the **Giry monad**, which maps X to the space of probability distributions over the Borel algebra $(X, B(X))$, with the so-called "weak topology".

Objective

We want to generalize our finite recurrence theorem:

Theorem (finite recurrence)

If (X, γ) is a strongly connected finite Markov chain and $E \subseteq X$ a non-empty measurable set, then reaching E infinitely often (starting from anywhere) is almost certain, i.e.

$$X \vDash \mathbb{P}(\Box \Diamond E) = 1.$$

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If (X, γ) is a strongly connected ~~finite~~ **topological** Markov chain and $E \subseteq X$ a non-empty measurable set, then reaching E infinitely often (starting from anywhere) is almost certain, i.e.

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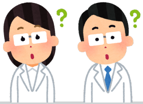

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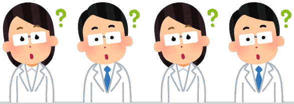

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1. What is the $\mathbb{P}(\Box \Diamond U)$ at a state $x \in X$? i.e. how to define the probability to follow a random walk that satisfies $\Box \Diamond U$?
2. What does "strongly connected" means?

Path spaces

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3. if $n = \infty$,

$$X^{\odot \infty} = \lim \left(\cdots \longrightarrow X^{\odot n} \longrightarrow X^{\odot (n-1)} \longrightarrow \cdots \longrightarrow X^{\odot 2} \longrightarrow X \right)$$

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1. $X^{\odot n} \subseteq X^n$ is the subspace of all tuples $(x_1, \ldots, x_n)$ such that x_{i+1} is reachable in one step from x_i;

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1. $X^{\odot n} \subseteq X^n$ is the subspace of all tuples $(x_1, \ldots, x_n)$ such that x_{i+1} is reachable in one step from x_i;
2. $X^{\odot\infty} \subseteq X^\infty$ is the subspace of all sequences $(x_i)_{i\in\mathbb{N}}$ such that x_{i+1} is reachable in one step from x_i;

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The path space $X^{\odot n}$ is Polish, for $n \leq \infty$.

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Let's sketch the proof. Since $\mathcal{P}\mathrm{ol}$ has countable limits, it is enough to show that $X^{\odot 2}$ is a Polish space.

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Recall that $X^{\odot 2} \subseteq X^2$ is the subspace of all pairs (x, y) such that $y \in \operatorname{supp} \gamma(x)$.

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Problem

$g_U : X^2 \longrightarrow \mathbb{R}$ is not continuous in general.

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Using some topological arguments, we get

$$X^{\odot 2} = \bigcap_{q,n} \tilde{g}_{q,n}^{-1}(0,+\infty).$$

Proposition

The Borel algebra of $X^{\odot n}$ is generated by sets of the form

$$E_1 \odot \cdots \odot E_n := X^{\odot n} \cap (E_1 \times \cdots \times E_n)$$

called **sequence sets**, where $E_1, \ldots, E_n \in B(X)$.

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The Borel algebra of $X^{\odot\infty}$ is generated by sets of the form

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In plain words, $\mathrm{Cyl}(E_1, \dots, E_n)$ is the set of all random walks that start in E_1, then go in E_2, then E_3, then ... then E_n, and then are free to go wherever they want.

Extension of probability measures

Motivation

Thanks to $X^{\odot n}$, we have a good notion of reachability and paths, a.k.a. random walks.

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But what is the probability to follow a random walk within a subset $E \subseteq X^{\odot\infty}$?

Objective

We need to lift an initial distribution μ on X to a distribution $\mathrm{ext}_\infty\, \mu$ on $X^{\odot\infty}$.

Extension of probabilities

Given a probability distribution μ on X (that acts as an initial distribution), we define a distribution $\mathrm{ext}_\infty \mu$ on $X^{\odot\infty}$ as follows: the probability

$$\mathrm{ext}_\infty \mu\,(\mathsf{Cyl}(E_1, \ldots, E_n))$$

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But we are not interested in any V, we would like

$$\Diamond\, U, \qquad \Box\,\Diamond\, U$$

where $U \subseteq X$ is open.

Logic

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1. **Linear temporal logic (LTL)** to express properties about random walks;
2. **Probabilistic computational tree logic (PCTL)** to express properties about states.

Linear Temporal Logic (LTL)

LTL is defined as

$$\phi \quad ::= \quad \top \;\mid\; \neg\phi \;\mid\; \phi \wedge \phi \;\mid\; E \;\mid\; \bigcirc \phi \;\mid\; \phi \,\mathbf{U}^{\leq n}\, \phi$$

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The semantic of LTL

The two modalities we're really interested in are

1. ("eventually") $\Diamond\,\phi := \top\,\mathbf{U}^{\leq\infty}\,\phi$:

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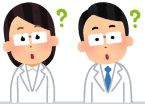
The semantic of LTL

Observation

The LTL formula we're really interested in is $\Box \Diamond U$:

$$[\![\Box \Diamond U]\!] = \left\{ (x_i)_{i \in \mathbb{N}} \in X^{\odot \infty} \mid x_i \in U \text{ for infinitely many } i \in \mathbb{N} \right\}.$$

Recurrence theorem



Theorem??

If (X, γ) is a strongly connected ~~finite~~ **topological** Markov chain and $U \subseteq X$ a non-empty **open** set, then reaching U infinitely often (starting from anywhere) is almost certain, i.e.

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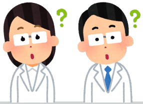

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$\Box \Diamond U =$ "it is always true that eventually, we'll reach U"

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$$\Phi \quad ::= \quad \top \;\mid\; \neg\Phi \;\mid\; \Phi \wedge \Phi \;\mid\; E \;\mid\; \mathbb{P}(\phi) \bowtie p$$

where $E \in B(X)$, ϕ is an LTL formula, $\bowtie \in \{>, \geq, \leq, <, =, \neq\}$, and $p \in [0, 1]$.

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Instead of expressing properties about **random walks** (like LTL), PCTL expresses properties about **states**:

$$\Phi \quad ::= \quad \top \mid \neg\Phi \mid \Phi \wedge \Phi \mid E \mid \mathbb{P}(\phi) \bowtie p$$

where $E \in B(X)$, ϕ is an LTL formula, $\bowtie \in \{>, \geq, \leq, <, =, \neq\}$, and $p \in [0, 1]$. The semantic of a PCTL formula Φ is a measurable set $[\![\Phi]\!] \subseteq X$:

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$$[\![\mathbb{P}(\phi) \geq p]\!] := \{x \in X \mid \mathrm{ext}_\infty\, \delta_x([\![\phi]\!]) \geq p\}$$

The semantic of PCTL

Quick dissection:

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Recurrence theorem

Theorem??

If (X, γ) is a strongly connected ~~finite~~ **topological** Markov chain and $U \subseteq X$ a non-empty **open** set, then reaching U infinitely often (starting from anywhere) is almost certain, i.e.

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The recurrence theorem(s)

Recurrence theorem: first attempt

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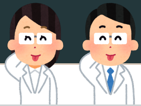

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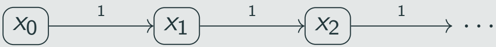
Counterexample



$\mathbb{P}(\Box \Diamond x_0) = 0$ everywhere...

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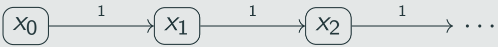

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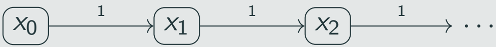

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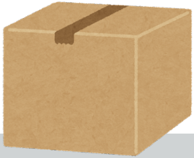
So it seems finiteness, or smallness, is important to prevent random walks from straying forever.

Compact Markov chains



Nugget of wisdom 2

$$\text{Topology} + \text{“smallness”} = \text{compactness}$$

We say that a Markov chain (X, γ) is **compact** if X is a compact topological space.

Recurrence theorem: second attempt

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If (X, γ) is a strongly connected ~~finite~~ **topological** Markov chain and $U \subseteq X$ a non-empty **open** set, then reaching U infinitely often (starting from anywhere) is almost certain, i.e.

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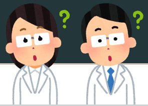

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$\mathbb{P}(\Box \Diamond x_0) = 0$ everywhere...

But x_0 was not reachable from everywhere in the first place, so this counterexample seems a bit unfair...

Recurrence theorem: second attempt

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If (X, γ) is a [redacted] **compact topological** Markov chain and $U \subseteq X$ a non-empty **open** set, then reaching U infinitely often (starting from anywhere) is almost certain, i.e.

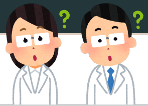
$$X \models \mathbb{P}(\Box \Diamond U) = 1.$$

Counterexample

$\mathbb{P}(\Box \Diamond x_0) = 0$ everywhere...

But x_0 was not reachable from everywhere in the first place, so this counterexample seems a bit unfair... After all, $\mathbb{P}(\Box \Diamond x_1) = 1$ everywhere.

Recurrence theorem: weak version



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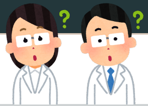

Theorem??

Let (X, γ) be a [redacted] **compact topological** Markov chain and $U \subseteq X$ a non-empty **open** set. If U is reachable from everywhere with probability > 0, i.e. $X \models \mathbb{P}(\Diamond U) > 0$, then U is almost certainly reached infinitely often from everywhere, i.e.

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Recurrence theorem: weak version

Theorem (weak recurrence) ✓

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What about "strong connectedness"?

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Recall that "strong connectedness" means (in the finite discrete case): every state is reachable from every other.

Strong connectedness?

Key observation

If (X, γ) is a (finite and discrete) strongly connected Markov chain, i.e. if every state is reachable from every other, then surely there cannot exist a proper **subchain** $(Y, \gamma|_Y) \subseteq (X, \gamma)$. Random walks in Y could possibly "escape" outside of Y.

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Definition

A topological Markov chain is *irreducible* if it does not have any proper subchains.

Irreducible Markov chains

Example

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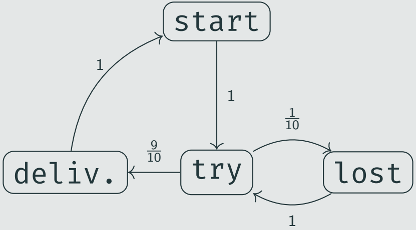
… is not irreducible.

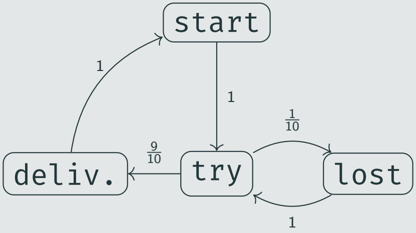
Irreducible Markov chains

Example



Irreducible Markov chains

Example



Irreducible Markov chains

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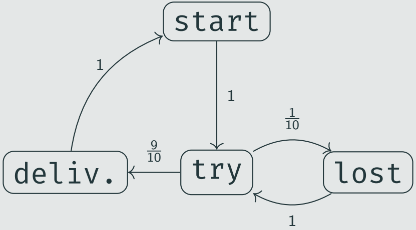

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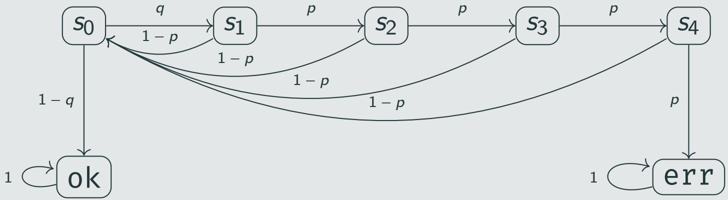

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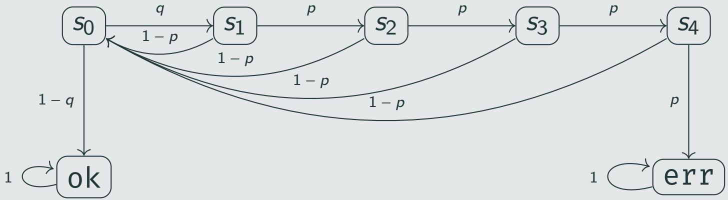

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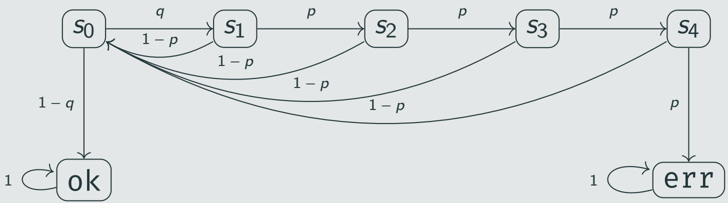

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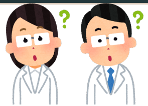
Recurrence theorem: strong version



This intuition leads to our second recurrence theorem

Theorem??

Let (X, γ) be a strongly connected **compact topological** Markov chain and $U \subseteq X$ an **open** set. If U is reachable from everywhere with probability > 0, i.e. $X \vDash \mathbb{P}(\lozenge U) > 0$, then U is almost certainly reached infinitely often from everywhere, i.e.

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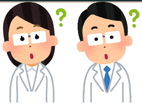

Theorem??

Let (X, γ) be a **irreducible compact topological** Markov chain and $U \subseteq X$ an **open** set. If $X \vDash \mathbb{P}(\Diamond U) > 0$, i.e. U is reachable from everywhere with probability > 0, then U is almost certainly reached infinitely often from everywhere, i.e.

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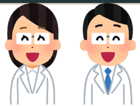

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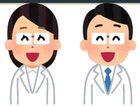

Theorem (strong recurrence, but better) ✓

Let (X, γ) be a **irreducible compact topological** Markov chain and $U \subseteq X$ a non-empty **open** set. [redacted] then U is almost certainly reached infinitely often from everywhere, i.e.

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If (X, γ) is a strongly connected finite Markov chain and $x \in X$ is a state, then x is almost certainly reached infinitely often from everywhere, i.e. $X \vDash \mathbb{P}(\Box \Diamond x) = 1$.

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Next steps

Question 1

Does every Markov chain necessarily admit an irreducible subchain?

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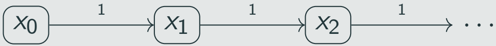

What about **compact** chains?

Question 2

If so, does every random walk necessarily end in an irreducible subchain?

$$X \vDash \mathbb{P}\left(\Diamond \bigcup_{Y \text{ irred.}} Y \right) = ?$$

Next steps

Similar recurrence results are known in the field of dynamical systems.

Poincarré's recurrence theorem

Let X be a measurable space, $\mu \in \Delta X$, $f : X \longrightarrow X$ be measure preserving (i.e. $\mu = \mu f^{-1}$), and $U \subseteq X$ be such that $\mu(U) > 0$. For almost all $x \in U$, $\mathbb{P}(\Box \Diamond U) = 1$.

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Similar recurrence results are known in the field of dynamical systems.

Poincarré's recurrence theorem

Let X be a measurable space, $\mu \in \Delta X$, $f\colon X \longrightarrow X$ be measure preserving (i.e. $\mu = \mu f^{-1}$), and $U \subseteq X$ be such that $\mu(U) > 0$. For almost all $x \in U$, $\mathbb{P}(\Box \Diamond U) = 1$.

Investigate connections between dynamical systems and Markov chains? How does our recurrence theorems transfer? How does the logical side transfer?

Next steps

1. Topological bisimulations

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2. Links with Büchi automata?

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2. Links with Büchi automata?
3. "Higher Markov chains" using simplicial sets. Geometrical detection of recurrence phenomena?

Now for some gory details

Where is the difficulty coming from?

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Where is the difficulty coming from? The subtle structure of the **Giry monad** $\Delta : \mathcal{P}\mathrm{ol} \longrightarrow \mathcal{P}\mathrm{ol}$.

The Giry monad

There are two Giry monads

$$\Delta : \mathcal{M}\mathrm{eas} \longrightarrow \mathcal{M}\mathrm{eas}, \qquad \Delta : \mathcal{P}\mathrm{ol} \longrightarrow \mathcal{P}\mathrm{ol}$$

The Giry monad on $\mathcal{M}\mathrm{eas}$

The "measurable Giry monad" $\Delta : \mathcal{M}\mathrm{eas} \longrightarrow \mathcal{M}\mathrm{eas}$ is fairly simple: for $X = (X, \Sigma) \in \mathcal{M}\mathrm{eas}$

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1. ΔX is the set of all probability measures on X
2. ... with the coarsest σ-algebra Σ_Δ such that for all $U \in \Sigma$, the map

$$\begin{aligned} I_U : \Delta X &\longrightarrow \mathbb{R} \\ \mu &\longmapsto \mu(U) = \int \chi_U \, \mathrm{d}\mu \end{aligned}$$

   is measurable.

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In other words, Σ_Δ is the *coarsest σ-algebra w.r.t. evaluation of probability measures at measurable sets*.

The unit $\delta : X \longrightarrow \Delta X$ maps x to its Dirac distribution δ_x.

Lemma

Equivalently, Σ_Δ is the coarsest σ-algebra such that for all measurable and bounded map $f\colon X \longrightarrow \mathbb{R}$, the map

$$\begin{aligned} I_f : \Delta X &\longrightarrow \mathbb{R} \\ \mu &\longmapsto \int f \,\mathrm{d}\mu \end{aligned}$$

is measurable. In other words, Σ_Δ is the *coarsest σ-algebra w.r.t. integration of measurable maps*.

The Giry monad on $\mathcal{P}\mathrm{ol}$, the wrong way

Let $X = (X, \mathcal{T})$ now be a Polish space. A naive generalization would be

Wrong definition

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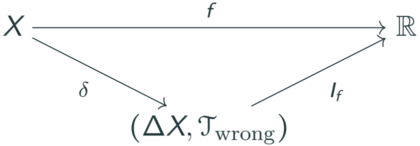
We also need $\delta : X \longrightarrow \Delta X$ to be continuous if we want a monad $\Delta : \mathcal{P}\mathrm{ol} \longrightarrow \mathcal{P}\mathrm{ol}$.

The Giry monad on $\mathcal{P}\mathrm{ol}$, the wrong way

But here's the problem, for any measurable $f\colon X \longrightarrow \mathbb{R}$, the following triangle commutes:



so X has the property that every measurable map is continuous, which forces X to be discrete...

The Giry monad on $\mathcal{P}\mathrm{ol}$

Wrong definition

ΔX is the set of probability measures on $(X, B(X))$ with the coarsest topology $\mathcal{T}_{\text{wrong}}$ such that for all measurable and bounded map $f\colon X \longrightarrow \mathbb{R}$, the map

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The Giry monad on $\mathcal{P}\mathrm{ol}$

Correct definition

ΔX is the set of probability measures on $(X, B(X))$ with the coarsest topology $\mathcal{T}_\Delta$ such that for all ~~measurable~~ **continuous** and bounded map $f\colon X \longrightarrow \mathbb{R}$, the map

$$\begin{aligned} I_f\colon \Delta X &\longrightarrow \mathbb{R} \\ \mu &\longmapsto \int f \,\mathrm{d}\mu \end{aligned}$$

is continuous.

The Giry monad on $\mathcal{P}\mathrm{ol}$

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The Giry monad on $\mathcal{P}\mathrm{ol}$

Why is this definition such a problem? $\mathcal{T}_{\mathrm{wrong}}$ is simple, but the only control we have over $\mathcal{T}_{\Delta}$ is

Theorem (Portmanteau)

For $\mu, \mu_0, \mu_1, \mu_2, \ldots \in \Delta X$, the following are equivalent:

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Proving that a map $\Delta X \longrightarrow Y$ is continuous is hard!

Extension of probability measures

For example, we were unable to prove that

$$\mathrm{ext}_n : \Delta X \longrightarrow \Delta X^{\odot n}$$

is continuous, where $2 \leq n \leq \infty$.

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is continuous, where $2 \leq n \leq \infty$. Still,

Proposition

$\mathrm{ext}_n : \Delta X \longrightarrow \Delta X^{\odot n}$ is **measurable**, for $n \leq \infty$.

Extension of probability measures

The cases $n = 0, 1$ are trivial (recall $X^{\odot 0} = *$, $X^{\odot 1} = X$).

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Fact

The Borel algebra of $\Delta(X^{\odot n})$ is generated by

$$\beta^{\bowtie p}(E_1 \odot \cdots \odot E_n) := \left\{ \nu \in \Delta(X^{\odot n}) \mid \nu(E_1 \odot \cdots \odot E_n) \bowtie p \right\}$$

where $\bowtie \in \{<, \leq, \geq, >\}$, $p \in [0, 1]$, and $E_1, \ldots, E_n \in B(X)$.

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Corollary

It is enough to show that

$$\mathrm{ext}_n{}^{-1}\left(\beta^{\bowtie p}(E_1 \odot \cdots \odot E_n)\right)$$

is measurable in ΔX.

Extension of probability measures

Recall that for $\mu \in \Delta X$,

$$\operatorname{ext}_n \mu(E_1 \odot \cdots \odot E_n) = \int_{x_1 \in E_1} \int_{x_2 \in E_2} \cdots \int_{x_{n-1} \in E_{n-1}} \mu(\mathrm{d}x_1) \times \gamma(x_1, \mathrm{d}x_2) \times \cdots \times \gamma(x_{n-1}, E_n).$$

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by induction, f is bounded measurable! So integrating w.r.t. to f is a measurable operation, i.e. $\mathrm{ext}_n\, \mu(E_1 \odot \cdots \odot E_n)$ is measurable in μ.

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Extension of probability measures

Now for the case $n = \infty$. By definition,

$$\text{ext}_\infty \mu\,(\text{Cyl}(E_1, \ldots, E_k)) = \text{ext}_k \mu(E_1 \odot \cdots \odot E_k)$$

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Extension of probability measures

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$$\mathrm{ext}_\infty\, \mu\,(\mathrm{Cyl}(E_1, \dots, E_k)) = \mathrm{ext}_k\, \mu(E_1 \odot \cdots \odot E_k)$$

which is measurable in μ. Since $\mathrm{ext}_\infty(-)\,(\mathrm{Cyl}(E_1, \dots, E_k))$ is measurable for every cylinder set $\mathrm{Cyl}(E_1, \dots, E_k)$, we conclude that $\mathrm{ext}_\infty(-)$ is measurable.

Extension of probability measures

To summarize, for $n \leq \infty$

$$\mathrm{ext}_n : \Delta X \longrightarrow \Delta X^{\odot n}$$

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Extension of probability measures

To summarize, for $n \leq \infty$

$$\mathrm{ext}_n : \Delta X \longrightarrow \Delta X^{\odot n}$$

is measurable.

BUT we do not know if it is continuous and we will have to work around that...

The weak recurrence theorem

We want to prove

Theorem (weak recurrence)

Let (X, γ) be a compact topological Markov chain and $U \subseteq X$ an open set. If $X \vDash \mathbb{P}(\Diamond U) > 0$, then

$$X \vDash \mathbb{P}(\Box \Diamond U) = 1.$$

for the sake of exposition, we will work backwards.

The weak recurrence theorem

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The weak recurrence theorem

Theorem (weak ~~recurrence~~ reachability)

Let (X, γ) be a compact topological Markov chain and $U \subseteq X$ an open set. If $X = [\![\mathbb{P}(\Diamond U) > 0]\!]$, then

$$X = [\![\mathbb{P}(\Diamond U) = 1]\!] .$$

If U is reached from everywhere with probability 1, then surely it is reached infinitely often.

The weak recurrence theorem

Theorem ("reachability soon")

Let (X, γ) be a compact topological Markov chain and $U \subseteq X$ an open set. If $X = [\![\mathbb{P}(\Diamond U) > 0]\!]$, then

$$X = \left[\!\left[\mathbb{P}(\Diamond^{\leq k} U) > r\right]\!\right]$$

for some k and r.

If U is reached soon ($\leq k$) with probability $> r$, then surely, avoiding U forever is impossible, i.e.

$$X = \left[\!\left[\mathbb{P}(\Diamond^{\leq k} U) > r\right]\!\right] \iff X = [\![\mathbb{P}(\Diamond U) = 1]\!] .$$

The weak recurrence theorem

So we need to show that there exist $k \in \mathbb{N}$ and $r \in [0,1]$ such that

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Torwards a contradiction, assume that for all $k, n \in \mathbb{N}$,

$$R_{k,n} := [\![\mathbb{P}(\Diamond^{\leq k} U) \leq 1/n]\!] \neq \varnothing$$

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So we need to show that there exist $k \in \mathbb{N}$ and $r \in [0, 1]$ such that

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Since X is compact, this implies that one $R_{k,n}$ is empty, thus proving that

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BUT we need to know that the $R_{k,n}$'s are **closed**!

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$$R_{k,n} = \llbracket \mathbb{P}(\Box^{\leq k} \bar{U}) \geq 1 - 1/n \rrbracket$$

$$= \left\{ x \in X \mid \operatorname{ext}_{k+1} \delta_x(\bar{U} \odot \cdots \odot \bar{U}) \geq 1 - 1/n \right\}$$

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In other words,

$$R_{k,n} = \Upsilon_k^{-1}[1 - 1/n, 1]$$

where $\Upsilon_k : x \longmapsto \operatorname{ext}_{k+1} \delta_x(\bar{U} \odot \cdots \odot \bar{U})$.

The weak recurrence theorem

So we have $\Upsilon_k : x \longmapsto \operatorname{ext}_{k+1} \delta_x(\bar{U} \odot \cdots \odot \bar{U})$,

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This is where now knowing if ext_∞ is continuous is a problem.

But there is a way out: it is enough to show that Υ_k is **upper semicontinuous** (USC).

Definitions

A map $f\colon X \longrightarrow \mathbb{R}$ is USC if for all $r \in \mathbb{R}$, $f^{-1}[r, +\infty)$ is closed.

The weak recurrence theorem

So we prove that $\Upsilon_k : x \longmapsto \text{ext}_{k+1}\, \delta_x(\bar{U} \odot \cdots \odot \bar{U})$ is USC.

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If $k = 0$, then

$$\Upsilon_0(x) = \delta_x(\bar{U}) = \chi_{\bar{U}}(x)$$

is USC since $\bar{U}$ is closed.

The weak recurrence theorem

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The case $k \geq 2$ can be deduced from induction, so let's focus on $k = 1$.

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From weak to strong

Theorem (weak recurrence)

Let (X, γ) be a compact topological Markov chain and $U \subseteq X$ an open set. If $X \vDash \mathbb{P}(\Diamond U) > 0$, then

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Using weak recurrence, it is enough to show that $X \vDash \mathbb{P}(\lozenge U) > 0$.

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Using weak recurrence, it is enough to show that $X \vDash \mathbb{P}(\Diamond U) > 0$. But this proof is not very interesting or insightful so let's move on

Topological & pathological

Polish spaces are fairly well-behaved. But topological Markov chains are not. Let's go over some frustrating counterexamples.

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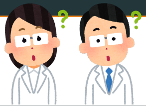

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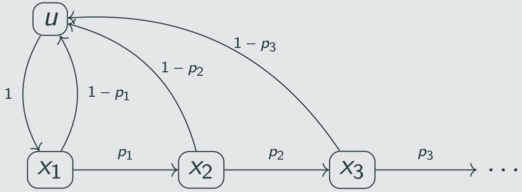
Counterexample

Consider the discrete chain



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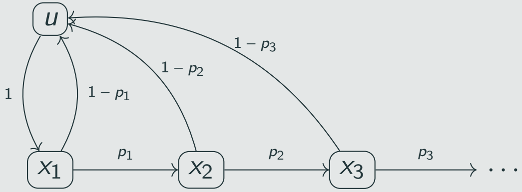

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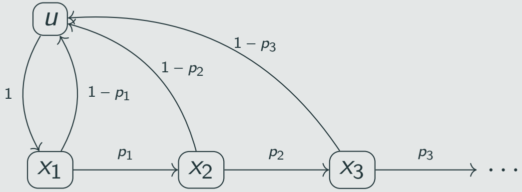

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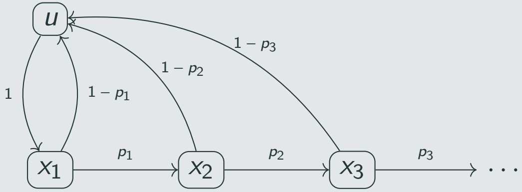

where $p_i := 1 - \frac{1}{(i+1)^2}$. u is reachable from everywhere, i.e. $X \vDash \mathbb{P}(\Diamond\{u\}) > 0$. Unfortunately,

$$\delta_{x_1}(\Diamond\{u\}) = 1 - \mathrm{ext}_\infty\, \delta_{x_1}(\{(x_1, x_2, \ldots)\}) = 1 - \prod_{i=1}^{\infty}\left(1 - \frac{1}{(i+1)^2}\right) = \frac{1}{2}.$$

Even with reachability, the compactness criterion is necessary!

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Simply consider $X = \mathbb{R}$ and $\gamma(x) = \delta_x$, i.e. every state can only transition to itself. Every subset is a subchain.

However

Lemma

If $Y \subseteq X$ is irreducible, then it is measurable.

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Therefore,

$$Y = \bigcup_{y \in Y} \operatorname{supp} \gamma(y) = \bigcup_{q \in Q} \operatorname{supp} \gamma(q)$$

is a countable union of closed sets.

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Y is not connected in general

Take $X := \{x_0, x_1\}$ with the discrete topology, and $\gamma(x_i) := \delta_{x_{1-i}}$. Then X itself is irreducible but not connected.

Reachability property

Definition

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In other words, X is “almost irreducible”.

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In other words, X is "almost irreducible". The weak recurrence theorem can be rephrased as follows:

Theorem (weak recurrence)

If X is compact and has the reachability property in itself, then for all nonempty open set U, $X \vDash \mathbb{P}(\Box \Diamond U) = 1$.

Reachability property

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A subchain $Y \subseteq X$ has the **reachability property** (in X) if for all nonempty open set $U \subseteq X$, we have $Y \vDash \mathbb{P}(\lozenge U) > 0$.

In other words, X is "almost irreducible". The weak recurrence theorem can be rephrased as follows:

Theorem (weak recurrence)

If X is compact and has the reachability property in itself, then for all nonempty open set U, $X \vDash \mathbb{P}(\Box \lozenge U) = 1$.

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The reachability property is an important notion to establish the strong recurrence theorem.

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Take $X := [0,1]_A + (0,1]_B$. The subscripts are purely decorative, if $x \in [0,1]$, let x_A be the corresponding element in the A component, and likewise for x_B (if $x > 0$).

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This Markov chain clearly has the reachability property, but it is not irreducible, for $Y := (0,1]_A + (0,1]_B$ is a proper subchain.

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1. The topological study of Markov chains is a bit tricky due to the nature of $\Delta : \mathcal{P}\mathrm{ol} \longrightarrow \mathcal{P}\mathrm{ol}$.
2. We saw how to walk around problems in the proof of the weak recurrence theorem.
3. We saw some basic counterexamples.