Algebraic K-Theory

Prof. K. Hess-Bellwald
TEXed by Cédric Ho Thanh

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Chapter 1

Introduction : motivations and relations to other fields

Algebraic K-theory can be viewed as homology theory for rings. It consists in a family of functors

$$K_n : \mathbf{Ring} \longrightarrow \mathbf{Ab}, \qquad \forall n \in \mathbb{N}$$

thar "behave like" homology of spaces. During this semester, we will study K_0 and K_1.

1.1 K_0

The idea is due to Groethendieck (1958) :

$$\begin{array}{ccccc} \mathbf{C} & \longrightarrow & \operatorname{Iso}\mathbf{C} & \longrightarrow & K(\mathbf{C}) \in \operatorname{Obj}\mathbf{Ab} \\ \text{a category} & & \text{isomorphism classes of } \mathbf{C} & & \text{Groethendieck group of } \mathbf{C} \end{array}.$$

We will apply this general construction to $\mathbf{C} = \mathscr{P}(R)$, the category of "nice" modules over the ring R. If R is a field, $\mathscr{P}(R)$ is the category of finite-dimensional vector space over R.

$$\mathscr{P}(R) \longrightarrow \operatorname{Iso}\mathscr{P}(R) \longrightarrow K(\operatorname{Iso}\mathscr{P}(R)) = K_0(R).$$

1.1.1 Motivation from linear algebra

Let $\mathbb{F}$ be a field and $\mathscr{V}_{\mathbb{F}}^{<\infty}$ be the category of finite-dimensional $\mathbb{F}$-vector space. We have a bijection

$$\begin{aligned} \dim_{\mathbb{F}} : \operatorname{Iso}\mathscr{V}_{\mathbb{F}}^{<\infty} &\longrightarrow \mathbb{N} \\ [V] &\longmapsto \dim_{\mathbb{F}} V \end{aligned}$$

since $V \cong W$ iff $\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} W$. Moreover :

$$\dim_{\mathbb{F}}(V \oplus W) = \dim_{\mathbb{F}} V + \dim_{\mathbb{F}} W$$
$$V \cong V',\ W \cong W' \implies V \oplus W \cong V' \oplus W'.$$

So :

- $\oplus$ induces a binary operation on $\operatorname{Iso}\mathscr{V}_{\mathbb{F}}^{<\infty}$: $[V]+[W]=[V\oplus W]$,
- $\dim_{\mathbb{F}}([V]+[W]) = \dim_{\mathbb{F}}[V] + \dim_{\mathbb{F}}[W]$.

But $(\mathbb{N},+)$ is not a group, and we'd rather work with groups.

$$\begin{tikzcd} \operatorname{Iso}\mathscr{V}_{\mathbb{F}}^{<\infty} \arrow[r, "\dim_{\mathbb{F}}"] \arrow[d, hook] & \mathbb{N} \arrow[d, hook] \\ K_0(\mathbb{F}) \arrow[r, "\exists!"] & \mathbb{Z} \end{tikzcd}.$$

1.1.2 Generalisation to arbitrary ring

Let $\mathbf{C}$ be the category of "nice" R-modules and A be an abelian group. A fonction $d : \operatorname{Iso}\mathbf{C} \longrightarrow A$ is a generalized rank (or dimension) if :

$$d([M]+[N]) = d([M]) + d([N]), \qquad \forall [M],[N] \in \operatorname{Iso}\mathbf{C}.$$

$K_0(R)$ is the target of the universal generalized rank, i.e. $\exists d_R : \operatorname{Iso}\mathscr{P}(R) \longrightarrow K_0(R)$ such that every other generalized rank $d : \operatorname{Iso}\mathscr{P}(R) \longrightarrow A$ factors uniquely through d_R :

$$\begin{tikzcd} \operatorname{Iso}\mathscr{P}(R) \arrow[r, "d_R"] \arrow[dr, "d"'] & K_0(R) \arrow[d, dashed, "\exists! f"] \\ & A \end{tikzcd}.$$

So $K_0(R)$ captures all "dimension type" information about R.

1.1.3 Relations to other subjects

- Number theory : Let R be a Dedekind domain (very nice commutative integral domain) and let $\operatorname{Cl}(R)$ be the ideal class group of R (measures how far R is for being a principal ideal domain). Then $K_0(R) = \operatorname{Cl}(R) \oplus \mathbb{Z}$.
- Representation theory : Let $\mathbb{F}$ be a field of characteristic 0 and G be a finite group. Consider the group algebra $\mathbb{F}[G]$. Then $K_0(\mathbb{F}[G]) = \operatorname{char}_{\mathbb{F}}(G)$, the character ring of G over $\mathbb{F}$, where an $\mathbb{F}$-character is a composition :

  $$G \xrightarrow{\ \rho\ } \mathrm{GL}_n(\mathbb{F}) \xrightarrow{\ \mathrm{tr}\ } \mathbb{F}.$$

  Notice that tr also preserves sums !
- Geometric topology : Let X be a connected topological space. Question : When does there exists a finite-dimensional CW-complex Y such that $X \simeq Y$? Awnser (Wall, 1965) : $\exists \widetilde{\chi} \in K_0(\mathbb{Z}[\pi_1 X])/\mathbb{Z}$, the finiteness obstruction, such that $\widetilde{\chi} = 0$ iff $X \simeq Y$ for a finite-dimensional CW-complex Y. This is a purely algebraic awnser to a topological problem !

"Douglas Adams said that the awnser is 42, maybe it's K_0."

Prof. K. Hess-Bellwald
21/02/2013

1.2 K_1

K_1 is motivated by the notion of determinant, a multiplicative invariant.

1.2.1 Motivation from linear algebra

Let $\mathbb{F}$ be a field. Then

$$\det : \mathrm{GL}_n(\mathbb{F}) \longrightarrow F^*$$

has the property that

$$\begin{aligned} \det(AB) &= \det A \cdot \det B, \\ \det(EA) &= \det A, \end{aligned}$$

where $A, B, E \in \mathrm{GL}_n(\mathbb{F})$ are matrices and E is an elementary matrix.

1.2.2 Generalisation to arbitrary ring

A generalized determinant consists in a group G and in a family of maps $\{\delta_n\}_{n\in\mathbb{N}^*}$ where $\delta_n : \mathrm{GL}_n(\mathbb{F}) \longrightarrow G$ satifies :

- the following diagram commutes :

$$\begin{tikzcd} \mathrm{GL}_n(\mathbb{F}) \arrow[rr, hook] \arrow[dr, "\delta_n"'] & & \mathrm{GL}_{n+1}(\mathbb{F}) \arrow[dl, "\delta_{n+1}"] \\ & G & \end{tikzcd}$$

  with the inclusion

$$\begin{aligned} \mathrm{GL}_n(\mathbb{F}) &\longrightarrow \mathrm{GL}_{n+1}(\mathbb{F}) \\ A &\longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

- $\delta_n(AB) = \delta_n(A)\delta_n(B)$,
- $\delta_n(E) = 1_G$, where E is an elementary matrix.

$K_1(R)$ is the target of the universal generalized determinant, i.e. $\exists \delta_n^R : \mathrm{GL}_n(R) \longrightarrow K_1(R)$ a generalized determinant such that every other genaralized determinant $\delta_n : \mathrm{GL}_n(R) \longrightarrow G$ factors uniquely through δ_n^R :

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, "\delta_n^R"] \arrow[dr, "\delta_n"'] & K_1(R) \arrow[d, dashed, "\exists! f_n"] \\ & G \end{tikzcd}.$$

So $K_1(R)$ captures all of the "determinant type" information about R.

1.2.3 Relations to other subjects

- Geometric topology : Let $f : X \longrightarrow Y$ be a homotopy equivalence of finite dimensional CW-complexes. Then f is a simple homotopy equivalence (given by composing particular elementary homotopy equivalences) if the Whitehead torsion of f : $\tau(f) \in K_1(\mathbb{Z}[\pi_1 Y])/\langle \pm 1, \pi_1 Y\rangle$ is 0. Another purely algebraic awnser to a topological question !

Chapter 2

K_0 and classification of modules

2.1 Definition and elementary properties of K_0

2.1.1 Group completion

2.1.1 Definition (Semigroup)**.** A **semigroup** consists of a set S together with an associative binary operation

$$\begin{aligned} S \times S &\longrightarrow S \\ (s, s') &\longmapsto s * s'. \end{aligned}$$

Homomorphisms are defined in the obvious way. The category of semigroups is written **SGrp** .

2.1.2 Examples.
1. Any group has an underlying semigroup. We have a forgatful functor $\mathscr{U} : \mathbf{Grp} \longrightarrow \mathbf{SGrp}$.
2. $(\mathbb{N}^*, +)$ and $(\mathbb{N}, \cdot)$.
3. $(\operatorname{Iso} \mathscr{V}_{\mathbb{F}}^{<\infty}, +)$.
4. Let X be a set. Then $(\mathscr{P}(X), \cap)$ is a semigroup.

2.1.3 Remark. $\dim_{\mathbb{F}} : (\operatorname{Iso} \mathscr{V}_{\mathbb{F}}^{<\infty}, +) \longrightarrow (\mathbb{N}, +)$ is a homomorphisme of semigroups.

How to turn a semigroup into an abelian group in a natural way ?

2.1.4 Definition (Group completion)**.** A **group completion** of a semigroup $(S, *)$ consists of an abelian group A together with a homomorphism of semigroups $f : S \longrightarrow \mathscr{U} A$ such that $\forall B \in \operatorname{Obj} \mathbf{Ab}$, every semigroup homomorphism $g : S \longrightarrow \mathscr{U} B$ factors uniquely through f :

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "g"'] & \mathscr{U} A \arrow[d, dashed, "\exists! \widehat{g}"] \\ & \mathscr{U} B \end{tikzcd}.$$

2.1.5 Remark. If the group completion of S exists, then it is unique up to isomorphism.

2.1.6 Definition (Free abelian group)**.** The **free abelian group functor** is given by :

$$
\begin{aligned}
F_{\mathbf{Ab}} : \mathbf{Set} &\longrightarrow \mathbf{Ab} \\
X &\longmapsto \bigoplus_{x\in X} \mathbb{Z}x \\
\left(X \xrightarrow{f} Y\right) &\longmapsto \left(F_{\mathbf{Ab}}X \xrightarrow{F_{\mathbf{Ab}}f} F_{\mathbf{Ab}}Y\right)
\end{aligned}
$$

where

$$
\begin{aligned}
F_{\mathbf{Ab}}f : F_{\mathbf{Ab}}X &\longrightarrow F_{\mathbf{Ab}}Y \\
\sum_{x\in X} m_x x &\longmapsto \sum_{x\in X} m_x f(x) = \sum_{y\in Y}\left(\sum_{x\in f^{-1}(y)} m_x\right) y.
\end{aligned}
$$

2.1.7 Remark. The functor $F_{\mathbf{Ab}}$ satisfies a universal property : $\forall A \in \operatorname{Obj}\mathbf{Ab}$ every set map $f : X \longrightarrow A$ factors uniquely through $\iota : X \hookrightarrow F_{\mathbf{Ab}}X$:

$$
\begin{tikzcd}
X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & F_{\mathbf{Ab}}X \arrow[d, dashed, "\exists!\widehat{f}"] \\
& A
\end{tikzcd}
.
$$

2.1.8 Theorem. There exists a functor $(-)^\wedge : \mathbf{SGrp} \longrightarrow \mathbf{Ab}$ such that $(S,*)^\wedge$ is the group completion of S.

Proof. We can define group completion by :

$$(S,*)^\wedge = F_{\mathbf{Ab}}S/\left\langle s*s' - s - s' \mid s,s' \in S\right\rangle.$$

For convenience, we note $D = \langle s*s' - s - s' \mid s,s' \in S\rangle$, so $(S,*)^\wedge = F_{\mathbf{Ab}}S/D$. We need to show the universal property. Let $(A,+)$ be an abelian group and $f : (S,*) \longrightarrow \mathscr{U}(A,+)$ be a semigroup homomorphism. Define

$$
\begin{aligned}
\gamma : (S,*) &\longrightarrow (S,*)^\wedge \\
s &\longmapsto [s].
\end{aligned}
$$

By the universal property of $F_{\mathbf{Ab}}$, $\exists!\widetilde{f} : F_{\mathbf{Ab}}S \longrightarrow (A,+)$ such that

$$
\begin{tikzcd}
S \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & F_{\mathbf{Ab}}S \arrow[d, dashed, "\widetilde{f}"] \\
& (A,+)
\end{tikzcd}
,
$$

i.e. $\widetilde{f}(s) = f(s), \forall s \in S$. Observe that we have

$$
\begin{tikzcd}
S \arrow[r, hook, "\iota"'] \arrow[rr, bend left, "\gamma"] \arrow[dr, "f"'] & F_{\mathbf{Ab}}S \arrow[r, two heads, "\pi"'] \arrow[d, dashed, "\exists!\widetilde{f}"] & (S,*)^\wedge \arrow[dl, dashed, "\exists!\widehat{f}"] \\
& (A,+) &
\end{tikzcd}
\iff \ker\pi = D \subseteq \ker\widetilde{f}
.
$$

We have

$$\begin{aligned}\widetilde{f}(s*s'-s-s') &= \widetilde{f}(s*s')-\widetilde{f}(s)-\widetilde{f}(s')\\ &= \underbrace{f(s*s')}_{=f(s)+f(s')} -f(s)-f(s')\\ &= 0.\end{aligned}$$

So $\ker\pi = D \subseteq \ker\widetilde{f}$ and $(S,*)^\wedge$ satisfies the universal property. ☐

2.1.9 Properties. 1. $\forall w \in (S,*)^\wedge$, $\exists s,t\in S$ such that $w=[s]-[t]$.

2. If $(S,*)$ is abelian, then

   (a) $[s]=[s'] \iff \exists u\in S$ such that $s*u=s'*u$,

   (b) $[s]-[t]=[s']-[t'] \iff \exists u\in S$ such that $s*t'*u=s'*t*u$.

Proof. 1. Let $w=\sum_{i=1}^n \mu_i[s_i]$ where $\mu_i\in\mathbb{Z}^*$ and $s_i\in S$. Let $\{i_1,\dots,i_k\}=\{i\mid \mu_i>0\}$ and $\{j_1,\dots,j_l\}=\{i\mid\mu_i<0\}$. Write

$$\begin{aligned}w &= \sum_{\nu=1}^k \mu_{i_\nu}[s_{i_\nu}]-\sum_{\nu=1}^l|\mu_{j_\nu}|[s_{j_\nu}]\\ &= \sum_{\nu=1}^k \left[s_{i_\nu}^{*\mu_{i_\nu}}\right]-\sum_{\nu=1}^l\left[s_{j_\nu}^{|\mu_{j_\nu}|}\right]\\ &= \left[s_{i_1}^{*\mu_{i_1}}*\cdots*s_{i_k}^{*\mu_{i_k}}\right]-\left[s_{j_1}^{*|\mu_{j_1}|}*\cdots*s_{j_l}^{*|\mu_{j_l}|}\right].\end{aligned}$$

2. (a)

$\Leftarrow$: One have

$$\begin{aligned}s*u=s'*u &\implies [s*u]=[s'*u]\\ &\implies [s]+[u]=[s']+[u]\\ &\implies [s]=[s'].\end{aligned}$$

$\Rightarrow$: One have that $[s]=[s'] \implies s-s'\in D=\langle t*t'-t-t'\mid t,t'\in S\rangle$, i.e. $\exists\mu_i\in\mathbb{Z}^*$, $\exists t_i,t_i'\in S$ such that

$$s-s'=\sum_{i=1}^n\mu_i(t_i*t_i'-t_i-t_i').$$

Let $\{i_1,\dots,i_k\}=\{i\mid\mu_i>0\}$ and $\{j_1,\dots,j_l\}=\{i\mid\mu_i<0\}$. Write

$$s+\sum_{\nu=1}^l|\mu_{j_\nu}|(t_{j_\nu}*t'_{j_\nu})+\sum_{\nu=1}^k\mu_{i_\nu}(t_{i_\nu}+t'_{i_\nu}) = s'+\sum_{\nu=1}^k\mu_{i_\nu}(t_{i_\nu}*t'_{i_\nu})+\sum_{\nu=1}^l|\mu_{j_\nu}|(t_{j_\nu}+t'_{j_\nu}).$$

This is an equation un $F_{\mathbf{Ab}}S$. It follows that in S

$$\begin{aligned}&s*(t_{j_1}*t'_{j_1})^{|\mu_{j_1}|}*\cdots*(t_{j_l}*t'_{j_l})^{|\mu_{j_l}|}*t_{i_1}^{\mu_{i_1}}*(t'_{i_1})^{\mu_{i_1}}*\cdots*t_{i_k}^{\mu_{i_k}}*(t'_{i_k})^{\mu_{i_k}}\\ &=s'*(t_{i_1}*t'_{i_1})^{|\mu_{i_1}|}*\cdots*(t_{i_k}*t'_{i_k})^{|\mu_{i_k}|}*t_{i_1}^{\mu_{j_1}}*(t'_{j_1})^{\mu_{j_1}}*\cdots*t_{j_l}^{\mu_{j_l}}*(t'_{j_l})^{\mu_{j_l}}.\end{aligned}$$

Since S is abelian, we have

$$\begin{aligned}
&(t_{j_1} * t'_{j_1})^{|\mu_{j_1}|} * \cdots * (t_{j_l} * t'_{j_l})^{|\mu_{j_l}|} * t_{i_1}^{\mu_{i_1}} * (t'_{i_1})^{\mu_{i_1}} * \cdots * t_{i_k}^{\mu_{i_k}} * (t'_{i_k})^{\mu_{i_k}} \\
&= (t_{i_1} * t'_{i_1})^{|\mu_{i_1}|} * \cdots * (t_{i_k} * t'_{i_k})^{|\mu_{i_k}|} * t_{i_1}^{\mu_{j_1}} * (t'_{j_1})^{\mu_{j_1}} * \cdots * t_{j_l}^{\mu_{j_l}} * (t'_{j_l})^{\mu_{j_l}} \\
&= u.
\end{aligned}$$

and therefore $s * u = s' * u$.

(b) We have

$$\begin{aligned}
[s] - [t] = [s'] - [t'] &\implies [s] + [t'] = [s'] + [t] \\
&\implies [s * t'] = [s' * t] \\
&\implies \exists u \in S \text{ such that } s * t' * u = s' * t * u.
\end{aligned}$$

$\square$

2.1.10 Examples. 0. $(\emptyset, *)^\wedge = (\{0\}, +)$. There is two ways to see this :

- $F_{\mathbf{Ab}}\emptyset = (\{0\}, +)$,
- use the universal proterty :

$$\begin{tikzcd}
(\emptyset, *) \arrow[r, "\exists!"] \arrow[dr, "\exists!"'] & (\emptyset, *)^\wedge \arrow[d, dashed, "\exists!"] \\
& (A, +)
\end{tikzcd}.$$

1. $S = \{s\}$, $\exists! * : S \times S \longrightarrow S : (s, s) \longmapsto s * s = s$. Then, $(\{s\}, *)^\wedge = (\{0\}, +)$ because in $(\{s\}, *)^\wedge$, $[s] = [s * s] = [s] + [s]$, whence $[s] = 0$.

2. More generaly, is $s * s = s$, $\forall s \in S$, then $(S, *)^\wedge = (\{0\}, +)$. For example, if X is a set, then $(\mathscr{P}(X), \cap)^\wedge = (\{0\}, +)$.

3. $(\mathbb{N}^*, +)^\wedge = (\mathbb{Z}, +)$.

4. $(\mathbb{N}^*, \cdot)^\wedge = (\mathbb{Q}^*_+, \cdot)$.

2.1.11 Remarks.

- $(S, *)^\wedge \cong (T, *)^\wedge$ does not implies $(S, *) \cong (T, *)$.
- $\gamma : (S, *) \longrightarrow (S, *)^\wedge$ is non necessarily injective.

2.1.2 Elementary module theory

See at `http://wiki.epfl.ch/alg-kthy-2013/documents/Elements_of_module_theory.pdf`.

2.1.3 Grothendieck groups

A construction closely related to group completion.

2.1.12 Definition (Grothendieck group)**.** Let R be a ring and $\mathscr{C}$ be a subcategory of ${}_R\mathbf{Mod}$ (left R-modules) such that $\operatorname{Iso}\mathscr{C}$ is a set and $0 \in \operatorname{Obj}\mathscr{C}$. Then, the **Grothendieck group** of $\mathscr{C}$ is defined as

$$K_0\mathscr{C} = F_{\mathbf{Ab}}(\operatorname{Iso}\mathscr{C})/E,$$

where

$$E = \langle M - L - N \mid \text{There exists a short exact sequence in } \mathscr{C} : 0 \to L \to M \to N \to 0 \rangle .$$

In other word, if $0 \to L \to M \to N \to 0$ is an exact sequence in $\mathscr{C}$, then $[M] = [L] + [N]$ in $K_0\mathscr{C}$.

2.1.13 Remark. If $\mathscr{C}$ is closed under direct sum $\oplus$, then $\forall L, N \in \operatorname{Obj}\mathscr{C}$, there exist a short exact sequence $0 \to L \to L \oplus N \to N \to 0$ and so $[L \oplus N] = [L] + [N]$.

The key universal property here can be formulated as follows :

2.1.14 Definition (Generalized rank)**.** Let $\mathscr{C}$ be as above. A **generalized rank** on $\mathscr{C}$ is a function

$$r : \operatorname{Iso}\mathscr{C} \longrightarrow (A, +)$$

where $(A, +)$ is an abelian group, such that $r(M) = r(L) + r(N)$ for every short exact sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}$.

2.1.15 Proposition. Les $\mathscr{C}$ be as above. There is a well defined function $d_{\mathscr{C}} : \operatorname{Iso}\mathscr{C} \longrightarrow K_0\mathscr{C}$ that is a generalized rank and universal, i.e. every other generalized rank $r : \operatorname{Iso}\mathscr{C} \longrightarrow (A, +)$ factors uniquely through $d_{\mathscr{C}}$:

$$\begin{tikzcd}
\operatorname{Iso}\mathscr{C} \arrow[r, "d_{\mathscr{C}}"] \arrow[dr, "r"'] & K_0\mathscr{C} \arrow[d, dashed, "\exists! \widehat{r}"] \\
& (A,+)
\end{tikzcd}.$$

Proof. Let $d_{\mathscr{C}}$ be the following compsite :

$$\operatorname{Iso}\mathscr{C} \xrightarrow{\ \iota\ } F_{\mathbf{Ab}}(\operatorname{Iso}\mathscr{C}) \xrightarrow{\ \pi\ } F_{\mathbf{Ab}}(\operatorname{Iso}\mathscr{C})/E = K_0\mathscr{C}.$$

It's a generalized rank because for all short exact sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}$,

$$\begin{aligned} d_{\mathscr{C}}(M) &= [M] \\ &= [L] + [N] \\ &= d_{\mathscr{C}}(L) + d_{\mathscr{C}}(N). \end{aligned}$$

It's universal because for every generalized rank $r : \operatorname{Iso}\mathscr{C} \longrightarrow (A, +)$, we have

$$\begin{tikzcd}
\operatorname{Iso}\mathscr{C} \arrow[r, hook, "\iota"'] \arrow[rr, bend left, "d_{\mathscr{C}}"] \arrow[dr, "r"'] & F_{\mathbf{Ab}}(\operatorname{Iso}\mathscr{C}) \arrow[r, two heads, "\pi"'] \arrow[d, dashed, "\exists! \widetilde{r}"] & K_0\mathscr{C} \arrow[dl, dashed, "\exists! \widehat{r} \iff \ker\pi = E \subseteq \ker\widetilde{r}"] \\
& (A,+) &
\end{tikzcd}.$$

Suppose $0 \to L \to M \to N \to 0$ in $\mathscr{C}$. Then

$$\begin{aligned}\widetilde{r}(M - L - N) &= \widetilde{r}(M) - \widetilde{r}(L) - \widetilde{r}(N)\\ &= r(M) - r(L) - r(N)\\ &= r(L + N) - r(L) - r(N)\\ &= 0.\end{aligned}$$

□

Relationship with the group completion :

2.1.16 Proposition. Let $\mathscr{C}$ be as above and suppose that $\forall L, N \in \operatorname{Obj}\mathscr{C}$ we have $L \oplus N \in \operatorname{Obj}\mathscr{C}$ (i.e. $\mathscr{C}$ is stable under direct sum $\oplus$) and that $L \hookrightarrow L \oplus N$, $L \oplus N \to N$ are morphisms of $\mathscr{C}$. Then there exists a surjective homomorphism $(\operatorname{Iso}\mathscr{C}, \oplus)^\wedge \longrightarrow K_0\mathscr{C}$.

Proof. $(\operatorname{Iso}\mathscr{C}, \oplus)^\wedge = F_{\mathbf{Ab}}(\operatorname{Iso}\mathscr{C})/D$ where $D = \langle L \oplus N - L - N \mid L, N \in \operatorname{Obj}\mathscr{C}\rangle$. So $D \leq E$ since there exists a short exact sequence $0 \to L \to L \oplus N \to N \to 0$. Therefore, there exists a surjective homomorphism

$$(\operatorname{Iso}\mathscr{C}, \oplus)^\wedge = F_{\mathbf{Ab}}(\operatorname{Iso}\mathscr{C})/D \longrightarrow F_{\mathbf{Ab}}(\operatorname{Iso}\mathscr{C})/E = K_0\mathscr{C}.$$

□

2.1.17 Examples.

1. Let $\mathscr{C} = \mathscr{S}(R)$, the full subcategory of finitely generated simple R-modules. If M is simple and is $0 \to L \xrightarrow{j} M \to N \to 0$ is exact, then, since j is injective, one has $L \cong j(L)$ which is a submodule of M. Therefore, either

   - $j(L) = 0$ which implies $L = 0$ and $M \cong N$:
   $$0 \to 0 \to M \xrightarrow{\cong} N \to 0,$$
   - $j(L) = M$ which implies $M \cong M$ and $N = 0$:
   $$0 \to L \xrightarrow{\cong} M \to 0 \to 0.$$

   So $E = \langle [M] - [M] \mid M \in \operatorname{Iso}\mathscr{C}\rangle = 0$ and
   $$K_0\mathscr{S}(R) = F_{\mathbf{Ab}}(\operatorname{Iso}\mathscr{S}(R)).$$

2. Let R be a ring and $\mathscr{F}(R)$ be the full subcategory of free and finitely generated left R-modules. If R has a **invariant basis number** (or **IBN**) (i.e. two isomorphic free modules have the same basis cardinality), then we can calculate $K_0\mathscr{F}(R)$. Any free module has a well defined rank : if X is a basis of M, then $\operatorname{rank} M = \sharp X$. In fact, we even have a well defined function
   $$\begin{aligned}\operatorname{rank} : \operatorname{Iso}\mathscr{F}(R) &\longrightarrow \mathbb{Z}\\ [M] &\longmapsto \operatorname{rank} M.\end{aligned}$$

   This is an example of a generalized rank. Given $0 \to L \to M \to N \to 0$, an exact sequence in $\mathscr{F}(R)$. It splits since N is free, so $M \cong L \oplus N$. If L is free of basis X and N of basis Y, then M is free on basis $X \amalg Y$. Si $\operatorname{rank} M = \operatorname{rank} L + \operatorname{rank} N$, and it is a generalized rank. Consequently :

$$\begin{tikzcd} \operatorname{Iso} \mathscr{F}(R) \arrow[r, "d_{\mathscr{F}(R)}"] \arrow[dr, "\text{rank}"'] & K_0\mathscr{F}(R) \arrow[d, "\exists! \widehat{r}"] \\ & \mathbb{Z} \end{tikzcd}.$$

In fact, $\widehat{r}$ is an isomorphism :

Surjectivity : If $n \geq 0$, then $\operatorname{rank} R^{\oplus n} = n$ and $\widehat{r}(-[R^{\oplus n}]) = -n$.

Injectivity : R has an IBN.

rom exercise set 3, if R is commutative, then it has a IBN. So, if R is commutative, then $K_0\mathscr{F}(R) \cong \mathbb{Z}$.

3. Let $\mathscr{P}(R)$ be the full subcategory of finitely generated projective left R-modules, and note $K_0 R = K_0\mathscr{P}(R)$ (the 0th algebraic theory group of R). Then
   - $\mathscr{P}(R)$ is closed under $\oplus$. If P and Q are projective, then $\exists P', Q'$ such that $P \oplus P'$ and $Q \oplus Q'$ are free. Then $(P \oplus P') \oplus (Q \oplus Q')$ is also free and $P \oplus Q$ is projective.
   - The homomorphism $(\operatorname{Iso} \mathscr{P}(R), \oplus)^\wedge \longrightarrow K_0\mathscr{P}(R)$ is an isomorphism. If $0 \to P \to P' \to P'' \to 0$ is an exact sequence in $\mathscr{P}(R)$, then is splits and $P' \cong P \oplus P''$.
4. Let $\mathscr{M}(R)$ be the full subcategory of finitely generated left R-modules, and note $G_0 R = K_0\mathscr{M}(R)$. Usually, $G_0 R \ncong (\operatorname{Iso} \mathscr{M}(R), \oplus)^\wedge$. For instance, if $R = \mathbb{Z}$, then we have an exact sequence

   $$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0 \qquad (p \text{ prime}),$$

   and therefore, in $G_0 R$, we have $[\mathbb{Z}] = [\mathbb{Z}] + [\mathbb{Z}/p\mathbb{Z}]$ and so $[\mathbb{Z}/p\mathbb{Z}] = 0$. However, in $(\operatorname{Iso} \mathscr{M}(R), \oplus)^\wedge$, we have $[\mathbb{Z}/p\mathbb{Z}] \neq 0$ since $\forall A$ an finitely generated abelian group, $A \ncong A \oplus \mathbb{Z}/p\mathbb{Z}$.

2.1.18 Remark. Why do we emphasize on the finite generation ? Because of the Eilenberg Swindle : if $\mathscr{C}$ is a subcategory of ${}_R\mathbf{Mod}$ closed under countable direct sum, then $K_0\mathscr{C} \cong \{0\}$. Indeed, let $M \in \operatorname{Obj} \mathscr{C}$. Then $N = \oplus_{i \in \mathbb{N}} M \in \operatorname{Obj} \mathscr{C}$. But $M \oplus N \cong N$ and so $[M] + [N] = [N]$ which implies $[M] = 0$.

2.1.4 Dévissage

The group $K_0\mathscr{C}$ can be very hard to compute ! We need tools to help with the computation. The first we'll see is the Dévissage.

2.1.19 Definition (Filtration)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be two subcategories of ${}_R\mathbf{Mod}$ such that $\mathscr{D}$ is a subcategory of $\mathscr{C}$. A **$\mathscr{D}$-filtration** of an object M in $\mathscr{C}$ is a sequence

$$\{0\} = M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$$

such that $M_i/M_{i+1} \in \operatorname{Obj} \mathscr{D}$, $\forall 0 \leq i < n$.

The idea is that if there exists a $\mathscr{D}$-filtration in $\mathscr{C}$ of M, then M is "build out of" objects of $\mathscr{D}$:

- $M_{n-1} \in \operatorname{Obj} \mathscr{D}$ since $M_n = \{0\}$,
- we have

$$0 \longrightarrow M_{n-1} \longrightarrow M_{n-2} \longrightarrow M_{n-2}/M_{n-1} \longrightarrow 0$$

  where M_{n-1} and M_{n-2}/M_{n-1} are objects of $\mathscr{D}$.

2.1.20 Lemma. Let $\{0\} = M_n \subseteq \cdots \subseteq M_0 = M$ be a $\mathscr{C}$ filtration in $\mathscr{C}$ of $M \in \operatorname{Obj}\mathscr{C}$. Then in $K_0\mathscr{C}$:

$$[M] = \sum_{i=0}^{n-1} [M_i/M_{i+1}].$$

Proof. We have

$$\begin{aligned}
\sum_{i=0}^{n-1} [M_i/M_{i+1}] &= \sum_{i=0}^{n-1} [M_i] - [M_{i+1}] \qquad \text{since } 0 \to M_{i+1} \to M_i \to M_i/M_{i+1} \to 0 \\
&= [M_0] - [M_n] \\
&= [M].
\end{aligned}$$

□

2.1.21 Lemma (Zassenhaus)**.** Given $M' \subseteq M$ and $N' \subseteq N$ in ${}_R\mathbf{Mod}$, then

$$\frac{M' + M \cap N}{M' + M \cap N'} \cong \frac{M \cap N}{M \cap N' + M' \cap N} \cong \frac{N' + M \cap N}{N' + M' \cap N}.$$

Proof, sketch. Define

$$\begin{aligned}
\phi : M' + M \cap N &\longrightarrow \frac{M \cap N}{M \cap N' + M' \cap N} \\
x' + x &\longmapsto [x].
\end{aligned}$$

- It is well defined because

$$x' + x = y' + y \iff x' - y = y' - x \in M' \cap (M \cap N) = M' \cap N.$$

- It is clearly a surjective homomorphism.
- $\ker \phi = M' + M \cap N'$.

□

2.1.22 Theorem (Dévissage)**.** Let $\mathscr{C}$ be a full subcategory of ${}_R\mathbf{Mod}$ and $\mathscr{D}$ a subcategory of $\mathscr{C}$, such that $0 \in \operatorname{Obj}\mathscr{D} \subseteq \operatorname{Obj}\mathscr{C}$ and $\operatorname{Iso}\mathscr{C}$, $\operatorname{Iso}\mathscr{D}$ are both sets. If

1. for all exact sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}$ we have

$$M \in \operatorname{Obj}\mathscr{D} \implies L, N \in \operatorname{Obj}\mathscr{D},$$

2. every object of $\mathscr{C}$ admit a $\mathscr{D}$-filtration in $\mathscr{C}$,

then

$$K_0\mathscr{C} \cong K_0\mathscr{D}.$$

Proof. Let us define two homomorphisms $K_0\mathscr{D} \longrightarrow K_0\mathscr{C}$ and $K_0\mathscr{C} \longrightarrow K_0\mathscr{D}$ that are mutually inverse.

- The inclusion functor $\iota : \mathscr{D} \longrightarrow \mathscr{C}$ induces a set map $\iota : \operatorname{Iso}\mathscr{D} \longrightarrow \mathscr{C}$. Consider

$$\begin{tikzcd} \operatorname{Iso}\mathscr{D} \arrow[r, "d_\mathscr{D}"] \arrow[d, "\iota"'] \arrow[dr, "d_\mathscr{C}\circ\iota"] & K_0\mathscr{D} \arrow[d, dashed, "\exists!\widehat{\iota}"] \\ \operatorname{Iso}\mathscr{C} \arrow[r, "d_\mathscr{C}"] & K_0\mathscr{C} \end{tikzcd},$$

  the map $d_\mathscr{C}\circ\iota$ is a generalized rank, so there exists a unique homomorphism $\widehat{\iota} : K_0\mathscr{D} \longrightarrow K_0\mathscr{C}$. Moreover, $\widehat{\iota}([M]_\mathscr{D}) = [M]_\mathscr{C}$.

- Let us define the inverse homomorphism. Inspired by lemma 2.1.20, define

$$\begin{aligned} r : \operatorname{Iso}\mathscr{C} &\longrightarrow K_0\mathscr{D} \\ M &\longmapsto \sum_{i=0}^{n-1} [M_i/M_{i+1}]_\mathscr{D}, \end{aligned}$$

  where $(M_i)_{i\le n} = (M_\bullet)$ is a $\mathscr{D}$-filtration of M in $\mathscr{C}$.

Now we prove the following results.

- $r(M)$ is independent of the chosen filtration (i.e. r is well defined). Let $M \in \operatorname{Obj}\mathscr{C}$ and suppose that $(M_i)_{i\le m}$ and $(N_i)_{i\le n}$ are two $\mathscr{D}$-filtrations of M in $\mathscr{C}$. We want to show that

$$\sum_{i=0}^{m-1} [M_i/M_{i+1}]_\mathscr{D} = \sum_{i=0}^{n-1} [N_i/N_{i+1}]_\mathscr{D}.$$

  We apply a technique used in the Schreier Refinment theorem, that is, build two new filtrations of M out of $(M_i)_{i\le m}$ and $(N_i)_{i\le n}$ that are refinenments, i.e. filters further between each consecutive pair of objetcs in $(M_i)_{i\le m}$ and $(N_i)_{i\le n}$, to end up with two filtrations of M of the same length and with the same quotients. To construct the refinments, define

$$\begin{aligned} M_{i,j} &= M_{i+1} + M_i \cap N_j \subseteq M_i \\ N_{i,j} &= N_{j+1} + M_i \cap N_j \qquad\qquad \forall 0 \le i \le m, \forall 0 \le j \le n. \end{aligned}$$

  On particular,

$$\begin{aligned} M_{i,0} &= M_{i+1} + M_i \cap N_0 = M_i \\ M_{i,n} &= M_{i+1} + M_i \cap N_n = M_{i+1}. \end{aligned}$$

  We then have a filtration :

$$M_{i+1} = M_{i,n} \subseteq M_{i,n-1} \subseteq \cdots \subseteq M_{i,1} \subseteq M_{i,0} = M_i.$$

  Similarily, we have another filtration :

$$N_{j+1} = N_{m,j} \subseteq \cdots \subseteq N_{0,j} = N_j.$$

We define the following two filtrations :

$$\begin{aligned} 0 = M_{m,n} \subseteq \cdots \subseteq M_{m,0} = M_{m-1} = M_{m-1,n} \subseteq \cdots \subseteq M_{0,0} = M \\ 0 = N_{m,n} \subseteq \cdots \subseteq N_{0,n} = N_{n-1} = N_{m,n-1} \subseteq \cdots \subseteq N_{0,0} = N, \end{aligned}$$

both of length mn. We will respectively note them $(M'_\bullet)$ and $(N'_\bullet)$. We need to show that these are $\mathscr{D}$-filtrations and compare the quotients. We have :

$$\begin{aligned} \frac{M_{i,j}}{M_{i,j+1}} &= \frac{M_{i+1} + M_i \cap N_j}{M_{i+1} + M_i \cap N_{j+1}} \\ &\cong \frac{M_i \cap N_j}{M_{i+1} \cap N_j + M_i \cap N_{j+1}} && \text{by Zassenhaus lemma} \\ &\cong \frac{N_{j+1} + M_i \cap N_j}{N_{j+1} + M_{i+1} \cap N_j} && \text{again, by Zassenhaus lemma} \\ &= \frac{N_{i,j}}{N_{i+1,j}}. \end{aligned}$$

So the quotients are the same. In particular, $(M'_\bullet)$ is a $\mathscr{D}$-filtration iff $(N'_\bullet)$ is. We need to show that

$$\frac{M_i \cap N_j}{M_{i+1} \cap N_j + M_i \cap N_{j+1}} \in \operatorname{Obj} \mathscr{D}.$$

By the 3rd isomorphism theorem, we have an exact sequence

$$0 \longrightarrow \frac{M_{i+1}+M_i\cap N_j}{M_{i+1}} \longrightarrow \frac{M_i}{M_{i+1}} \longrightarrow \frac{M_i}{M_{i+1}+M_i\cap N_j} \longrightarrow 0 \quad .$$

We have that $M_i/M_{i+1} \in \operatorname{Obj} \mathscr{D}$ (the middle term). So

$$\begin{aligned} &\frac{M_{i+1} + M_i \cap N_j}{M_{i+1}} \in \operatorname{Obj} \mathscr{D} \\ \implies &\frac{M_i \cap N_j}{M_{i+1} \cap N_j} \in \operatorname{Obj} \mathscr{D} && \text{by the 2nd isomorphism theorem.} \end{aligned}$$

No consider

$$0 \longrightarrow \frac{M_{i+1}\cap N_j+M_i\cap N_{j+1}}{M_{i+1}\cap N_j} \longrightarrow \frac{M_i\cap N_j}{M_{i+1}\cap N_j} \longrightarrow \frac{M_i\cap N_j}{M_{i+1}\cap N_j+M_i\cap N_{j+1}} \longrightarrow 0 \quad ,$$

by the 3rd isomorphism theorem. The middle term is an object of $\mathscr{D}$, so

$$\frac{M_{i+1} \cap N_j + M_i \cap N_{j+1}}{M_{i+1} \cap N_j} \in \operatorname{Obj} \mathscr{D}.$$

To conclude,

$$\begin{aligned} \sum_{i=0}^{m-1} [M_i/M_{i+1}]_{\mathscr{D}} &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} [M_{i,j}/M_{i,j+1}]_{\mathscr{D}} && \text{refinments doesn't change the sum} \\ &= \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} [N_{i,j}/N_{i+1,j}]_{\mathscr{D}} \\ &= \sum_{j=0}^{n-1} [N_j/N_{j+1}]_{\mathscr{D}}. \end{aligned}$$

So r is well defined.

- r is a generalized rank. Let

$$0 \longrightarrow L \xrightarrow{j} M \xrightarrow{p} N \longrightarrow 0$$

be a short exact sequence in $\mathscr{C}$. To show that $r(M) = r(L) + r(N)$, we build a $\mathscr{D}$-filtration of M in $\mathscr{C}$ out of $\mathscr{D}$-filtrations of L and N. Let $(L_i)_{i\leq l}$ and $(N_i)_{i\leq n}$ be such $\mathscr{D}$-filtrations, both in $\mathscr{C}$. Since $j(L_0) = j(L) = \operatorname{im} j = \ker p = p^{-1}(N_n)$, we have a filtration

$$0 = j(L_l) \subseteq j(L_{l-1}) \subseteq \cdots \subseteq j(L_0) = p^{-1}(N_n) \subseteq p^{-1}(N_{n-1}) \subseteq \cdots \subseteq p^{-1}(N_0) = M.$$

Observe that

  - $j(L_i)/j(L_{i+1}) \cong L_i/L_{i+1} \in \operatorname{Obj}\mathscr{D}$, since j is injective,
  - $p^{-1}(N_i)/p^{-1}(N_{i+1}) \cong N_i/N_{i+1} \in \operatorname{Obj}\mathscr{D}$.

So it is a $\mathscr{D}$ filtration. We have

$$\begin{aligned} r(M) &= \sum_{i=0}^{l-1}[j(L_i)/j(L_{i+1})]_{\mathscr{D}} + \sum_{i=0}^{n-1}[p^{-1}(N_i)/p^{-1}(N_{i+1})]_{\mathscr{D}} \\ &= \sum_{i=0}^{l-1}[L_i/L_{i+1}]_{\mathscr{D}} + \sum_{i=0}^{n-1}[N_i/N_{i+1}]_{\mathscr{D}} \\ &= r(L) + r(N). \end{aligned}$$

So r is a generalized rank.

We now have a unique induced homomorphism

$$\begin{tikzcd} \operatorname{Iso}\mathscr{C} \arrow[r, "d_{\mathscr{C}}"] \arrow[dr, "r"'] & K_0\mathscr{C} \arrow[d, dashed, "\exists!\widehat{r}"] \\ & K_0\mathscr{D} \end{tikzcd}.$$

Thus, $\widehat{r}([M]_{\mathscr{C}}) = r(M) = \sum_i [M_i/M_{i+1}]_{\mathscr{D}}$ for any $\mathscr{D}$-filtration $(M_\bullet)$. We have $\widehat{r} = \widehat{\iota}^{-1}$. Indeed :

- $\widehat{r}\circ\widehat{\iota} = \mathrm{id}_{K_0\mathscr{D}}$ since

$$\begin{aligned} \widehat{r}\circ\widehat{\iota}([M]_{\mathscr{D}}) &= \widehat{r}([M]_{\mathscr{C}}) \\ &= [M/0]_{\mathscr{D}} && \text{since } 0 \subset M \text{ is a } \mathscr{D}\text{-filtration of } M \\ &= [M]_{\mathscr{D}}. \end{aligned}$$

- Since $\forall M \in \operatorname{Obj}\mathscr{C}$, we can choose any $\mathscr{D}$-filtration $(M_\bullet)$ of M in $\mathscr{C}$ and calculate

$$\begin{aligned} \widehat{\iota}\circ\widehat{r}([M]_{\mathscr{C}}) &= \widehat{\iota}\left(\sum_i [M_i/M_{i+1}]_{\mathscr{D}}\right) \\ &= \sum_i \widehat{\iota}([M_i/M_{i+1}]_{\mathscr{D}}) \\ &= \sum_i [M_i/M_{i+1}]_{\mathscr{C}} \\ &= [M]_{\mathscr{C}} && \text{by lemma 2.1.20.} \end{aligned}$$

$\square$

2.1.23 Example. Let $R = \mathbb{Z}$, $\mathscr{C}$ be the full subcategory of finite abelian groups, and $\mathscr{D}$ be the full subcategory of cyclic groups of prime order, including the trivial group (i.e. the finitely generated simple left $\mathbb{Z}$-modules $\mathscr{F}(\mathbb{Z})$). Clearly, $\{0\} \in \operatorname{Obj}\mathscr{D} \subset \operatorname{Obj}\mathscr{C}$, and $\operatorname{Iso}\mathscr{D}$, $\operatorname{Iso}\mathscr{C}$ are both sets.

- By the same argument as before, if $0 \to L \to M \to N \to 0$ is exact in $\mathscr{D}$, then either L or N is trivial, and M is isomorphic to the other.
- By basic group theory, any finite abelian group admit a filtration by cyclic groups of prime order.

So, by Dévissage, $K_0\mathscr{C} \cong K_0\mathscr{D}$. Moreover, by previous example, we have

$$K_0\mathscr{D} = F_{\mathbf{Ab}}\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\} \cong F_{\mathbf{Ab}}\{x_p \mid p \text{ prime}\}.$$

Let's calculate this groupe another way. Recall that if $0 \to A \to B \to C \to 0$ is exact in $\mathscr{C}$, then $B/A \cong C$ and so $|B| = |A||C|$. We thus have a generalized rank

$$\begin{aligned} r : \operatorname{Iso}\mathscr{C} &\longrightarrow (\mathbb{Q}_+^*, \cdot) \\ A &\longmapsto |A|. \end{aligned}$$

By the universal property of $K_0\mathscr{C}$, we therefore have

$$\begin{tikzcd} \operatorname{Iso}\mathscr{C} \arrow[r, "d_{\mathscr{C}}"] \arrow[dr, "r"'] & K_0\mathscr{C} \arrow[d, dashed, "\exists!\widehat{r}"] \\ & (\mathbb{Q}_+^*, \cdot) \end{tikzcd}.$$

Claim : $\widehat{r}$ is an isomorphism.

Surjectivity : Consider $\frac{c}{d} \in \mathbb{Q}_+^*$. Then

$$\begin{aligned} \widehat{r}([\mathbb{Z}/c\mathbb{Z}] - [\mathbb{Z}/d\mathbb{Z}]) &= \widehat{r}([\mathbb{Z}/c\mathbb{Z}]) \cdot \widehat{r}([\mathbb{Z}/d\mathbb{Z}])^{-1} \\ &= r([\mathbb{Z}/c\mathbb{Z}]) \cdot r([\mathbb{Z}/d\mathbb{Z}])^{-1} \\ &= \frac{c}{d}. \end{aligned}$$

Injectivity : Argument by induction over the power of primes in $|A|$.

- If $|A| = p = |B|$ with p prime, then $A \cong \mathbb{Z}/p\mathbb{Z} \cong B$ and so $[A] = [B]$.
- Induction step : suppose that if $|A| = p^k = |B|$ with $k < n$, then $[A] = [B]$. Note that if $|A| = p^n$, then A has at least one element of order p, whence there exists a injective homomorphism $\mathbb{Z}/p\mathbb{Z} \longrightarrow A$, from which we can derive an exact sequence $0 \to \mathbb{Z}/p\mathbb{Z} \to A \to A' \to 0$, and $|A'| = p^{n-1}$. Similarily, we have an exact sequence $0 \to \mathbb{Z}/p\mathbb{Z} \to B \to B' \to 0$, and $|B'| = p^{n-1}$. By induction hypothesis, we have

$$\begin{aligned} [A] &= [\mathbb{Z}/p\mathbb{Z}] + [A'] \\ &= [\mathbb{Z}/p\mathbb{Z}] + [B'] \\ &= [B]. \end{aligned}$$

– For any abelian group A, if $|A| = p_1^{k_1} \cdot \cdots \cdot p_r^{k_r} = |B|$, then $A \cong A_1 \oplus \cdots \oplus A_r$, $B \cong B_1 \oplus \cdots \oplus B_r$, where $|A_i| = p_i^{k_i} = |B_i|$. So

$$\begin{aligned}[A] &= [A_1] + \cdots + [A_r] \\ &= [B_1] + \cdots + [B_r] \\ &= [B].\end{aligned}$$

2.1.5 The resolution theorem

Yet another tol for simplifying computations of K_0 by passing to a smaller, less complicated subcategory of ${}_R\mathbf{Mod}$. The key concept :

2.1.24 Definition (Projective resolution)**.** Let $M \in \mathrm{Obj}\, {}_R\mathbf{Mod}$. A **projective resolution** of M is an exact sequence in ${}_R\mathbf{Mod}$

$$\cdots \longrightarrow P_n \xrightarrow{p_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

where P_i is a projective module. The reolution is **finite** if there exists n such that $P_i = 0, \forall i > n$. If $P_i = 0, \forall i > n$ and $P_n \neq 0$, then the resolution has **length** n. The **projective dimension** of M is :

$$\mathrm{pd}\, M = \begin{cases} \infty & \text{if } M \text{ doesn't admit any finite projective resolution,} \\ n & \text{the minimum length of a finite projective resolution, if it exists.} \end{cases}$$

The **global dimension** of a ring R is

$$\mathrm{gldim}\, R = \sup_{M \in \mathrm{Obj}\, {}_R\mathbf{Mod}} \mathrm{pd}\, M.$$

Define $\mathscr{P}_{<\infty}(R)$, the full subcategory of $\mathscr{M}(R)$ whose objects are those that have a finite projective dimension.

2.1.25 Theorem (Resolution theorem)**.** For all ring R :

$$K_0\mathscr{P}_{<\infty}(R) \cong K_0R,$$

where $K_0R = K_0\mathscr{P}(R)$.

Proof. We need two homomorphisms $K_0R \longrightarrow K_0\mathscr{P}_{<\infty}(R)$, $K_0\mathscr{P}_{<\infty}(R) \longrightarrow K_0R$ that are inverse to each other. This is equivalent to the existance of generalized ranks $\mathrm{Iso}\,\mathscr{P}(R) \longrightarrow K_0\mathscr{P}_{<\infty}(R)$, $\mathrm{Iso}\,\mathscr{P}_{<\infty}(R) \longrightarrow K_0R$ such that the induced homomorphisms are mutually inverse. Let us note $[M]$ the class of a R-module M in K_0R and $[M]_\infty$ the class of M in $K_0\mathscr{P}_{<\infty}(R)$.

- Generalized rank $\mathrm{Iso}\,\mathscr{P}(R) \longrightarrow K_0\mathscr{P}_{<\infty}(R)$. Remark that $\mathscr{P}(R)$ is a subcategory of $\mathscr{P}_{<\infty}(R)$, si it makes sense to define

$$\begin{aligned}\iota : \mathrm{Iso}\,\mathscr{P}(R) &\longrightarrow K_0\mathscr{P}_{<\infty}(R) \\ M &\longmapsto [M]_\infty.\end{aligned}$$

  This is a generalized rank, since a sequence exact in $\mathscr{P}(R)$ is aslo exact in $\mathscr{P}_{<\infty}(R)$. So $[M]_\infty = [L]_\infty + [N]_\infty$ for all exact sequence $0 \to L \to M \to N \to 0$ in K_0R. Thus, there exists an induced homomorphism

$$\begin{aligned}\widehat{\iota} : K_0R &\longrightarrow K_0\mathscr{P}_{<\infty}(R) \\ [M] &\longmapsto [M]_\infty.\end{aligned}$$

- Generalized rank $\mathrm{Iso}\,\mathscr{P}_{<\infty}(R) \longrightarrow K_0R$. The **Euler characteristic** is the function

$$\begin{aligned}\chi_R : \mathrm{Iso}\,\mathscr{P}_{<\infty}(R) &\longrightarrow K_0R\\ M &\longmapsto \sum_{k=0}^{n}(-1)^k[P_k],\end{aligned}$$

where $0 \to P_n \to \cdots \to P_0 \to M \to 0$ is a projective resolution of M. We show two points.

1. χ_R is well defined, i.e. does not depend of the chosen projective resolution. Let

$$0 \to P_n \to \cdots \to P_0 \to M \to 0,$$

$$0 \to Q_n \to \cdots \to Q_0 \to M \to 0$$

be two projective resolutions of M (where it may be that some P_i, Q_j are 0). Let us show that

$$P_n \oplus Q_{n-1} \oplus P_{n-2} \oplus \cdots \oplus \begin{cases} P_0 & \text{if } 2|n \\ Q_0 & \text{otherwise}\end{cases} \cong Q_n \oplus P_{n-1} \oplus Q_{n-2} \oplus \cdots \oplus \begin{cases} Q_0 & \text{if } 2|n \\ P_0 & \text{otherwise}\end{cases}$$

$$\begin{tikzcd}
0 \arrow[r] & P_n \arrow[r] \arrow[dr, dotted, no head] & P_{n-1} \arrow[r] \arrow[dl, dotted, no head] \arrow[dr, dotted, no head] & P_{n-2} \arrow[r] \arrow[dl, dotted, no head] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[dr, dotted, no head] & P_0 \arrow[r] \arrow[dl, dotted, no head] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r] & Q_{n-2} \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}.$$

If $n = 1$, it is the Schanuel's Lemma. Suppose the isomorphism holds $\forall n \leq N$. Consider

$$\begin{tikzcd}
0 \arrow[r] & P_{N+1} \arrow[r, "p_{N+1}"] \arrow[d] & P_N \arrow[r, "p_N"] & P_{N-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & \ker p_N \arrow[ur, "p_{N+1}"'] & & & & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & Q_{N+1} \arrow[r, "q_{N+1}"] \arrow[d] & Q_N \arrow[r, "q_N"] & Q_{N-1} \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & \ker q_N \arrow[ur, "q_{N+1}"'] & & & & & & &
\end{tikzcd}.$$

The inductive hypothesis is that given K, L two modules and two exact sequences :

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[dr, dotted, no head] & P_n \arrow[r] \arrow[dl, dotted, no head] \arrow[dr, dotted, no head] & P_{n-1} \arrow[r] \arrow[dl, dotted, no head] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[dr, dotted, no head] & P_0 \arrow[r] \arrow[dl, dotted, no head] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd},$$

we have

$$K \oplus C_n \cong L \oplus D_n,$$

where $C_n = Q_n \oplus P_{n-1} \oplus Q_{n-2} \oplus \cdots$ and $D_n = P_n \oplus Q_{n-1} \oplus P_{n-2} \oplus \cdots$. It is true if $n = 0$ by Schanuel's Lemma. By induction hypothesis, $\ker p_N \oplus C_N \cong \ker q_N \oplus D_N$. Now we have an isomorphism

$$\begin{aligned} P_{N+1} \oplus C_N \cong \ker p_N \oplus C_N & \\ &= \ker q_N \oplus D_N \\ &= Q_{N+1} \oplus D_N. \end{aligned}$$

So $P_{N+1} \oplus C_N \cong Q_{N+1} \oplus D_N$. To conclude, we have toprove the induction hypothesis for $N+1$. Consider

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & P_{N+1} \arrow[r, "p_{N+1}"] \arrow[d] & P_N \arrow[r, "p_N"] & P_{N-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \\ & 0 \arrow[r] & \ker p_N \arrow[ur, "p_{N+1}"'] & & & & & & \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & Q_{N+1} \arrow[r, "q_{N+1}"] \arrow[d] & Q_N \arrow[r, "q_N"] & Q_{N-1} \arrow[r] & \cdots \arrow[r] & Q_0 \arrow[r] & M \arrow[r] & 0 \\ & 0 \arrow[r] & \ker q_N \arrow[ur, "q_{N+1}"'] & & & & & & \end{tikzcd}.$$

Again, by inductive hypothesis for N, we have $\ker p_N \oplus C_N \cong \ker q_N \oplus D_N$. Now we have short exact segences

$$0 \longrightarrow K \longrightarrow P_{N+1} \longrightarrow \ker p_N \longrightarrow 0$$

$$0 \longrightarrow L \longrightarrow Q_{N+1} \longrightarrow \ker q_N \longrightarrow 0,$$

whence

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & P_{N+1} \oplus C_N \arrow[r] & \ker p_N \oplus C_N \arrow[r] \arrow[d, leftrightarrow, "\cong"] & 0 \\ 0 \arrow[r] & L \arrow[r] & Q_{N+1} \oplus D_N \arrow[r] & \ker q_N \oplus D_N \arrow[r] & 0 \end{tikzcd}.$$

By Schanuel's Lemma, we have

$$\begin{aligned} K \oplus D_{N+1} &= K \oplus Q_{N+1} \oplus D_N \\ &\cong L \oplus P_{N+1} \oplus C_N \\ &= L \oplus C_{N+1}. \end{aligned}$$

Therefore, χ_R is well defined.

2. We need to show that for all exact sequence $0 \to L \to M \to N \to 0$ in $\mathscr{P}_{<\infty}(R)$, we have $\chi_R(M) = \chi_R(L) + \chi_R(N)$. By point 1., we can choose any projective resolution of L and N to calculate $\chi_R(L)$ and $\chi_R(N)$. So take

$$0 \to P_n \to \cdots \to P_0 \to L \to 0,$$

$$0 \to Q_n \to \cdots \to Q_0 \to N \to 0.$$

Apply the Horseshoe Lemma (Exercise set 5) :

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_0 \oplus Q_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}.$$

By point 1.,

$$\begin{aligned}
\chi_R(M) &= \sum_{k=0}^{n} (-1)^k [P_k \oplus Q_k] \\
&= \sum_{k=0}^{n} (-1)^k \left([P_k] + [Q_k]\right) \\
&= \chi_R(L) + \chi_R(N).
\end{aligned}$$

Finally, we have an induced homomorphism

$$\begin{aligned}
\widehat{\chi}_R : K_0\mathscr{P}_{<\infty}(R) &\longrightarrow K_0R \\
[M]_\infty &\longmapsto \sum_{k=0}^{n} (-1)^k [P_k],
\end{aligned}$$

where $0 \to P_n \to \cdots \to P_0 \to M \to 0$ is a projective resolution of M.

- We have that $\forall M \in \mathrm{Obj}\, \mathscr{P}(R)$,

$$\begin{aligned}
\widehat{\chi}_R \circ \widehat{\imath}([M]) &= \widehat{\chi}_R([M]_\infty) \\
&= [M] && \text{since } M \text{ is projective.}
\end{aligned}$$

So $\widehat{\chi}_R \circ \widehat{\imath} = \mathrm{id}_{\mathscr{P}(R)}$.

- Let $M \in \operatorname{Obj} \mathscr{P}_{<\infty}(R)$ and let $0 \to P_n \to \cdots \to P_0 \to M \to 0$ be a projective resolution of M. Then
$$\widehat{\iota} \circ \widehat{\chi}_R([M]_\infty) = \sum_{k=0}^{n} (-1)^k [P_k]_\infty.$$
Consider
$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[dr] & & 0 & & & & \\
 & & & \substack{\mathrm{im}\, p_2 \\ = \ker p_1 \\ = K_1} \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[dr] & \cdots \arrow[r] & P_2 \arrow[ur] \arrow[rr] & & P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[r] & M \arrow[r] & 0 \\
 & \substack{\mathrm{im}\, p_3 \\ = \ker p_2 \\ = K_2} \arrow[ur] \arrow[dr] & & & & \substack{\mathrm{im}\, p_1 \\ = \ker \epsilon \\ = K_0} \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}.$$
Remark that $\operatorname{pd} K_i < \infty$, $\forall 0 \le i \le n-1$. So we have a projective resolution
$$0 \to P_n \to \cdots \to P_{i+1} \to K_i \to 0.$$
So all objects in the big diagram above are in $\mathscr{P}_{<\infty}(R)$. For all i, we have an exact sequence $0 \to K_i \to P_i \to K_{i-1} \to 0$. So
$$\begin{aligned} [P_i]_\infty &= [K_i]_\infty + [K_{i-1}]_\infty \\ [P_n]_\infty &= [K_{n-1}]_\infty \\ [P_0]_\infty &= [K_0]_\infty + [M]_\infty. \end{aligned}$$
So
$$\widehat{\chi}_R([M]_\infty) = \sum_{k=0}^{n} (-1)^k [P_k]_\infty = [M]_\infty$$
and $\widehat{\chi}_R \circ \widehat{\iota} = \mathrm{id}_{\mathscr{P}(R)}$.

□

2.1.26 Corollary. If $\operatorname{gldim} R < \infty$, then $G_0 R = K_0 R$, where $G_0 R = K_0 \mathscr{M}(R)$.

Proof. We have :
$$\begin{aligned} G_0 R &= K_0 \mathscr{M}(R) \\ &= K_0 \mathscr{P}_{<\infty}(R) && \text{since } \operatorname{gldim} R < \infty \\ &\cong K_0 \mathscr{P}(R) \\ &= K_0 R. \end{aligned}$$

□

2.1.27 Examples. 0. $\mathrm{gldim}\, R = 0$ iff R is **semi simple** , i.e every R module is projective.

1. $\mathrm{gldim}\, R = 1$ iff every left ideal if R is projective. For instance, $\mathrm{gldim}\, \mathbb{Z}[\sqrt{-5}] = 1$.

2.1.6 Stability

The goal here is to give a different caracterisation of $[M]$ in K_0R as a type of equivalence class. Recall that

- $K_0R \cong (\mathrm{Iso}\, \mathscr{P}(R), \oplus)^\wedge$,
- if $(S, *)$ is an abelian semigroup, then in $(S, *)^\wedge$

$$[s] = [t] \iff \exists u \in S \text{ such that } s * u = t * u.$$

Apply this to K_0R to get

$$[M] = [N] \iff \exists P \in \mathrm{Obj}\, \mathscr{P}(R) \text{ such that } M \oplus P \cong N \oplus P.$$

Since P is projective, $\exists Q \in \mathrm{Obj}\, \mathscr{P}(R)$ such that $P \oplus Q \cong R^{\oplus n}$ is free. Therefore

$$[M] = [N] \iff \exists n \geq 0 \text{ such that } M \oplus R^{\oplus n} \cong N \oplus R^{\oplus n}.$$

2.1.28 Definitions (Stably isomorphic, equivalent, free)**.** Let $M, N \in \mathrm{Obj}\, {}_R\mathbf{Mod}$. We say that

- M and N are **stably isomorphic** is there exists $n \in \mathbb{N}$ such that $M \oplus R^{\oplus n} \cong N \oplus R^{\oplus n}$, and we note $M \cong_S N$,
- M and N are **stably equivalent** is there exists $n, m \in \mathbb{N}$ such that $M \oplus R^{\oplus n} \cong N \oplus R^{\oplus m}$, and we note $M \sim_S N$,
- M is **stably free** if $M \sim_S 0$, in other words, if there exists $n, m \in \mathbb{N}$ such that $M \oplus R^{\oplus n} \cong R^{\oplus m}$.

Why is it important for K_0 ?

2.1.29 Proposition. 1. Every element of K_0R is of the form $[P] - [R^{\oplus n}]$, for some $P \in \mathrm{Obj}\, \mathscr{P}(R)$, $n \in \mathbb{N}$.

2. In K_0R,
$$[P] = [Q] \iff P \cong_S Q.$$

3. In $\widetilde{K}_0R = K_0R/\langle [R] \rangle$, the **projective class group** ,
$$[[P]] = [[Q]] \iff P \sim_S Q.$$

Proof. 1. Recall that $K_0R = (\mathrm{Iso}\, \mathscr{P}(R), \oplus)^\wedge$, and that in a group completion $(S, *)^\wedge$, any element is of the form $[s] - [t]$, for some $s, t \in S$. So any element of K_0R is of the form $[P] - [Q]$, for some projective modules P and Q. Since Q is projective, there exists another projective module Q' sich that $Q \oplus Q' = R^{\oplus n}$. So

$$\begin{aligned} [P] - [Q] &= [P] + [Q'] - ([Q] + [Q']) \\ &= [P \oplus Q'] - [R^{\oplus n}], \end{aligned}$$

and $P \oplus Q'$ is projective.

2. Already done.
3. $P \sim_S Q$ iff there exists $m, n \in \mathbb{N}$ such that $P \oplus R^{\oplus m} \cong Q \oplus R^{\oplus n}$. Without loss of generality, $m \leq n$. So

$$\begin{aligned} P \sim_S Q &\iff P \cong_S Q \oplus R^{\oplus(n-m)} \\ &\iff [P] = [Q \oplus R^{\oplus(n-m)}] = [Q] + [R^{\oplus(n-m)}] = [Q] + (n-m)[R] \qquad \text{by 2.} \\ &\implies [[P]] = [[Q]]. \end{aligned}$$

□

2.1.30 Notation. Let $\mathscr{F}^{\mathrm{st}}(R)$ be the full subcategory of the finitely generated stably free left R modules.

2.1.31 Remark. We have $\mathscr{F}(R) \subsetneq \mathscr{F}^{\mathrm{st}}(R) \subsetneq \mathscr{P}(R)$. For example :

- This example is due to Kaplansky. Let $R = \mathbb{R}[X, Y, Z]/\langle X^2 + Y^2 + Z^2 - 1\rangle$, and let $q : \mathbb{R}[X, Y, Z] \longrightarrow R$ be the quothent homomorphism. Note $\overline{X} = q(X)$, $\overline{Y} = q(Y)$, $\overline{Z} = q(Z)$. Define the two matrices :

$$A = (\overline{X}\,\overline{Y}\,\overline{Z}), \qquad B = \begin{pmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{pmatrix}.$$

Define

$$\begin{aligned} p : R^{\oplus 3} &\longrightarrow R \\ v &\longmapsto Av, \\ s : R &\longrightarrow R^{\oplus 3} \\ r &\longmapsto Br. \end{aligned}$$

Then

$$\begin{aligned} p \circ s(r) &= ABr \\ &= (\overline{X}^2 + \overline{Y}^2 + \overline{Z}^2)r \\ &= 1 \cdot r \\ &= r \\ &\implies p \circ s = \mathrm{id}_R\,. \end{aligned}$$

Consider the following exact sequence that splits

$$0 \longrightarrow P = \ker p \longrightarrow R^{\oplus 3} \xrightarrow{\;p\;} R \longrightarrow 0 \qquad .$$

We have $P \oplus R \cong R^{\oplus 3}$, so P is stably free. We will show that P is not free. By contradiction, suppose it is. Since R is commutative, and therefore has an IBN, there exists an isomorphism $f : R^{\oplus 2} \xrightarrow{\cong} P$. Given such a f, there exists an isomorphism

$$\begin{aligned} \phi : R^{\oplus 3} &\longrightarrow P \oplus R \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} &\longmapsto f(a, b) + c. \end{aligned}$$

On the other hand, the previous exact sequence gives us an isomorphisme

$$\begin{aligned}\theta : P \oplus R &\longrightarrow R^{\oplus 3}\\ x + r &\longmapsto x + s(r).\end{aligned}$$

Consider $\theta \circ \phi : R^{\oplus 3} \longrightarrow R^{\oplus 3}$. It is an isomorphism represented by the following invertible matrix :

$$\begin{pmatrix} a_1 & b_1 & \overline{X} \\ a_2 & b_2 & \overline{Y} \\ a_3 & b_3 & \overline{Z} \end{pmatrix}$$

whose determinant is a unit $u \in R$. Wa have a matrix

$$C = \begin{pmatrix} u^{-1}a_1 & b_1 & \overline{X} \\ u^{-1}a_2 & b_2 & \overline{Y} \\ u^{-1}a_3 & b_3 & \overline{Z} \end{pmatrix}$$

of determinant 1. Consider $C^0(\mathbb{S}^2, \mathbb{R})$ the ring of continuous functions on $\mathbb{S}^2$ with value in $\mathbb{R}$, and

$$\begin{aligned}\psi : \mathbb{R}[X, Y, Z] &\longrightarrow C^0(\mathbb{S}^2, \mathbb{R})\\ X &\longmapsto \mathrm{proj}_1\\ Y &\longmapsto \mathrm{proj}_2\\ Z &\longmapsto \mathrm{proj}_3\,.\end{aligned}$$

Note that $\forall w \in \mathbb{S}^2$ we have $\mathrm{proj}_1(w)^2 + \mathrm{proj}_2(w)^2 + \mathrm{proj}_3(w)^2 = 1$, so $\mathrm{proj}_1^2 + \mathrm{proj}_2^2 + \mathrm{proj}_3^2 = 1$, and so

$$\begin{aligned}\psi(X^2 + Y^2 + Z^2 - 1) &= \psi(X)^2 + \psi(Y)^2 + \psi(Z)^2 - 1\\ &= 0\\ &\Longrightarrow \langle X^2 + Y^2 + Z^2 - 1\rangle \subseteq \ker \psi.\end{aligned}$$

Therefore, there exists a unique $\widehat{\psi} : R \longrightarrow C^0(\mathbb{S}^2, \mathbb{R})$ such that the following diagram commutes

$$\begin{tikzcd} \mathbb{R}[X,Y,Z] \arrow[r, "\psi"] \arrow[d, "q"'] & C^0(\mathbb{S}^2, \mathbb{R}) \\ R \arrow[ur, dashed, "\widehat{\psi}"'] & \end{tikzcd},$$

with $\widehat{\psi}(\overline{X}) = \mathrm{proj}_1$, $\widehat{\psi}(\overline{Y}) = \mathrm{proj}_2$, $\widehat{\psi}(\overline{Z}) = \mathrm{proj}_3$. Apply $\widehat{\psi}$ to C to get

$$D = \widehat{\psi}(C) = \begin{pmatrix} \widehat{\psi}(u^{-1}a_1) & \widehat{\psi}(b_1) & \mathrm{proj}_1 \\ \widehat{\psi}(u^{-1}a_2) & \widehat{\psi}(b_2) & \mathrm{proj}_2 \\ \widehat{\psi}(u^{-1}a_3) & \widehat{\psi}(b_3) & \mathrm{proj}_3 \end{pmatrix} \in M_3(C^0(\mathbb{S}^2, \mathbb{R})).$$

Since $\widehat{\psi}$ is a ring homomorphism, we have $\det D = \widehat{\psi}(\det C) = 1$. Let c_j be the j^{th} column vector. We have that $c_1(D) \wedge c_3(D) : \mathbb{S}^2 \longrightarrow \mathbb{R}^3$ is a continuous tangential vector field on $\mathbb{S}^2$. Moreover, it never vanishes since

$$|\det D| = |\langle c_1(D) \wedge c_3(D), c_2(D)\rangle| = 1.$$

But this is impossible by Brouwers Hairy Ball theorem. So P is not free, but stably free.

- Let $R = \mathbb{Z}/6\mathbb{Z}$. Consider the following exact sequence that splits :

$$0 \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/6\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \quad .$$

  The module $\mathbb{Z}/2\mathbb{Z}$ is therefore projective, but not stably free because $|\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/6\mathbb{Z})^n|$ is never a power of 6, and so $(\mathbb{Z}/6\mathbb{Z})^m \ncong \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/6\mathbb{Z})^n$, $\forall m, n \in \mathbb{N}$.

2.1.7 Multiplicative structure in K_0R

How to multiply classes of projective module ? The key notion :

2.1.32 Definition (Bilinear map)**.** Let R be a ring. Let $M \in \operatorname{Obj} \mathbf{Mod}_R$, $N \in \operatorname{Obj} {}_R\mathbf{Mod}$, and A be an abelian group. A **bilinear map** $f : M \times N \longrightarrow A$ is a function such that $\forall x, x' \in M$, $\forall y, y' \in N$, $\forall r \in R$:

1. $f(x + x', y) = f(x, y) + f(x', y)$,
2. $f(x, y + y') = f(x, y) + f(x, y')$,
3. $f(xr, y) = f(x, ry)$.

2.1.33 Theorem (Tensor product)**.** Let R be a ring. There is a functor

$$- \otimes_R - : \mathbf{Mod}_R \times {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$$

such that $\forall M \in \operatorname{Obj} \mathbf{Mod}_R$, $\forall N \in \operatorname{Obj} {}_R\mathbf{Mod}$, there exists a bilinear map

$$\eta : M \times N \longrightarrow M \otimes_R N$$

such that any other bilinear map $f : M \times N \longrightarrow A$ factors uniquely through η :

$$\begin{tikzcd} M \times N \arrow[r, "\eta"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, dashed, "\exists! \widehat{f}"] \\ & A \end{tikzcd} \quad .$$

The abelian group $M \otimes_R N$ is called the **tensor product** of M and N.

Proof. We give an explicit construction !

- Define
$$M \otimes_R N = F_{\mathbf{Ab}}(M \times N)/B,$$
  where B is the subgroup of $F_{\mathbf{Ab}}(M \times N)$ generated by
  - $(x + x', y) - (x, y) - (x', y)$, $\forall x, x' \in M$, $\forall y \in N$,
  - $(x, y + y') - (x, y) - (x, y')$, $\forall x \in M$, $\forall y, y' \in N$,
  - $(xr, y) - (x, ry)$, $\forall x \in M$, $\forall y \in N$, $\forall r \in R$.

Define η as the composite

$$\begin{tikzcd}
M \times N \arrow[r, hook] & F_{\mathbf{Ab}}(M \times N) \arrow[r, two heads] & M \otimes_R N \\
(x, y) \arrow[r, mapsto] & (x, y) \arrow[r, mapsto] & x \otimes y
\end{tikzcd}.$$

η is indeed bilinear. Given $f : M \times N \longrightarrow A$ a bilinear map, consider

$$\begin{tikzcd}
M \times N \arrow[r, hook] \arrow[rr, bend left, "\eta"] \arrow[dr, "f"'] & F_{\mathbf{Ab}}(M \times N) \arrow[r, two heads] \arrow[d, dashed, "\exists ! \widetilde{f}"] & M \otimes_R N \arrow[dl, dashed] & \exists ! \widehat{f} \iff B \subseteq \ker \widetilde{f} \\
& A & &
\end{tikzcd},$$

and $B \subseteq \ker \widetilde{f}$ since f is bilinear.

- Given $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$, define

$$\begin{aligned} f \otimes_R g : M \otimes_R N &\longrightarrow M' \otimes_R N' \\ x \otimes y &\longmapsto f(x) \otimes g(y). \end{aligned}$$

We have

$$\begin{tikzcd}
M \times N \arrow[r, hook] \arrow[d, "f \times g"'] \arrow[drr, "\text{bilinear}" description] & F_{\mathbf{Ab}}(M \times N) \arrow[r, two heads] & M \otimes_R N \arrow[d, dashed, "\exists !"] \\
M' \times N' \arrow[r, hook] & F_{\mathbf{Ab}}(M' \times N') \arrow[r, two heads] & M' \otimes_R N'
\end{tikzcd}.$$

This shows that $f \otimes_R g$ is well defined and that it is a homomorphism of groups.

$\square$

2.1.34 Corollary. Let R, S and T be rings. Then $- \otimes_S -$ restricts and corestricts to a functor :

$$- \otimes_S - : {}_R\mathbf{Mod}_S \times {}_S\mathbf{Mod}_T \longrightarrow {}_R\mathbf{Mod}_T.$$

Proof. Given $M \in \operatorname{Obj} {}_R\mathbf{Mod}_S$, $N \in \operatorname{Obj} {}_S\mathbf{Mod}_T$, define a left R-action on $M \otimes_S N$ by

$$\begin{aligned} \lambda : R \times (M \otimes_S N) &\longrightarrow M \otimes_S N \\ (r, x \otimes y) &= (rx) \otimes y, \end{aligned}$$

and a right T-action by

$$\begin{aligned} \rho : (M \otimes_S N) \times R &\longrightarrow M \otimes_S N \\ (x \otimes y, t) &= x \otimes (yt). \end{aligned}$$

We need to show that λ and ρ are well defined :

1. $r \cdot ((x + x') \otimes y) = r \cdot (x \otimes y + x' \otimes y)$,

2. $r \cdot (x \otimes (y + y')) = r \cdot (x \otimes y + x \otimes y')$,
3. $r \cdot ((x \cdot s) \otimes y) = r \cdot (x \otimes (s \cdot y))$.

$\square$

2.1.35 Remark. If R is commutative, and M is a left R-module, we can see M as a (R, R)-bimodule as follows :

$$x \cdot r = r \cdot x.$$

It is indeed a (R, R)-bimodule because R is commutative. We can then see $- \otimes_R -$ as a functor

$$- \otimes_R - : \mathbf{Mod}_R \times \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_R,$$
$$- \otimes_R - : {}_R\mathbf{Mod} \times {}_R\mathbf{Mod} \longrightarrow {}_R\mathbf{Mod}.$$

Here are some important properties of $- \otimes_R -$ seen in the exercises sets :

1. Associativity : if $L \in \operatorname{Obj} {}_Q\mathbf{Mod}_R$, $M \in \operatorname{Obj} {}_R\mathbf{Mod}_S$, $N \in {}_S\mathbf{Mod}_T$, then
$$(L \otimes_R M) \otimes_S N \cong L \otimes_R (M \otimes_S N)$$
as (Q, T)-bimodules.
2. Commutativity : if R is commutative, and $M, N \in \operatorname{Obj} \mathbf{Mod}_R$, then
$$M \otimes_R N \cong N \otimes_R M.$$
3. Additivity : if $M_i \in \operatorname{Obj} \mathbf{Mod}_R, \forall i \in I$ and $N_j \in \operatorname{Obj} {}_R\mathbf{Mod}, \forall j \in J$, then
$$\left(\bigoplus_{i \in I} M_i\right) \otimes_R \left(\bigoplus_{j \in J} N_j\right) \cong \bigoplus_{(i,j) \in I \times J} (M_i \otimes_R N_j).$$
4. Unit : if $M \in \operatorname{Obj} \mathbf{Mod}_R$ and $N \in \operatorname{Obj} {}_R\mathbf{Mod}$, then
$$M \otimes_R R \cong M, \qquad R \otimes_R N \cong N.$$
The idea of this proof is that $x \otimes r = x \otimes (r \cdot 1) = (x \cdot r) \otimes 1$.
5. If $M \in \operatorname{Obj} \mathbf{Mod}_R$ and $N \in \operatorname{Obj} {}_R\mathbf{Mod}$, then
$$M \otimes_R 0 \cong 0, \qquad 0 \otimes_R N \cong 0.$$
6. Projectives : if R is commutative and if P and Q are projectives right R-modules, then $P \otimes_R Q$ is also projective.

2.1.36 Definition (Semiring)**.** A **semiring** consists of an abelian semigroup $(S, +, 0)$ with a neutral element 0, endowed with an associative multiplication map $* : S \times S \longrightarrow S$ that has a unit 1 and that is distributive over the semigroup structure. It is commutative if $*$ is commutative.

2.1.37 Proposition. If R is commutative, then $(\operatorname{Iso} \mathscr{P}(R), \oplus, 0, \otimes_R, R)$ is a commutative semiring.

Proof. Obvious, with the previous properties. $\square$

2.1.38 Proposition. If $(S,+,0,*,1)$ is a (commutative) semiring, than $*$ induces a (commutative) ring structure on $(S,+,0)^\wedge$, i.e. we have a lifting of $(-)^\wedge : \mathbf{AbSGrp} \longrightarrow \mathbf{Ab}$:

$$\begin{tikzcd} \mathbf{AbSGrp} \arrow[r, "(-)^\wedge"] & \mathbf{Ab} \\ \mathbf{SRing} \arrow[u, "\mathscr{U}"] \arrow[r, "(-)^\wedge"] & \mathbf{Ring} \arrow[u, "\mathscr{U}"'] \end{tikzcd}.$$

Proof. Define $[s][t] = [s*t]$. We just have to show that this multiplication on $(S,+,0)^\wedge$ is well defined.

$$\begin{aligned} [s]=[s'] &\iff \exists u \in S \text{ such that } s+u = s'+u && \text{since } (S,+,0) \text{ is abelian} \\ &\implies \forall t \in S \quad (s+u)*t = (s'+u)*t \\ &\implies s*t+u*t = s'*t+u*t \\ &\implies [s*t] = [s'*t]. \end{aligned}$$

Similarly, $[t]=[t'] \implies [s*t]=[s*t']$. If $*$ is commutative, then $[s][t]=[s*t]=[t*s]=[t][s]$ and so $(S,+,0,*,1)^\wedge$ is commutative. $\square$

2.1.39 Corollary. If R is commutative, then $K_0R = (\operatorname{Iso}\mathscr{P}(R), \oplus, 0, \otimes_R, R)^\wedge$ is a commutative ring, where

$$[P][Q] = [P \otimes_R Q].$$

2.1.40 Example. If R is a commutative PID, then $K_0R \cong \mathbb{Z}$ as rings.

2.2 Functoriality of Grothendieck group

The goals here are :

- understand the relation between K_0R and R_0S with respect to a ring homomorphism $\phi : R \longrightarrow S$,
- give tools for computing K_0R from K_0R_i, where R is "built out of" the R_i's.

2.2.1 Exact functors

Whet you need to get a homomorphism between Grothendieck group :

2.2.1 Definition (Exact functor)**.** Let R and S be rings, and $\mathscr{C} \subseteq \mathbf{Mod}_R$ and $\mathscr{D} \subseteq \mathbf{Mod}_S$ be full subcategories with 0-object and only set of isomorphism classes of objects. A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is **exact** if it preserves exact sequences.

2.2.2 Proposition. If $F : \mathscr{C} \longrightarrow \mathscr{D}$ is exact, then it induces a homomorphism $K_0F : K_0\mathscr{C} \longrightarrow K_0\mathscr{D}$.

Proof. Since F preserves exact sequences, we have that $d_{\mathscr{D}} \circ F : \operatorname{Iso}\mathscr{C} \longrightarrow K_0\mathscr{D}$ is a generalized rank. By the universal property of $d_{\mathscr{C}}$, there exist a unique ring homomorphism $K_0F : K_0\mathscr{C} \longrightarrow K_0\mathscr{D}$:

$$\begin{tikzcd}
\operatorname{Iso}\mathscr{C} \arrow[r, "d_{\mathscr{C}}"] \arrow[d, "F"'] \arrow[dr, "d_{\mathscr{D}} \circ F"] & K_0\mathscr{C} \arrow[d, dashed, "\exists! K_0F"] \\
\operatorname{Iso}\mathscr{D} \arrow[r, "d_{\mathscr{D}}"] & K_0\mathscr{D}
\end{tikzcd}.$$

∎

"Exact functors are exactly what you need !"

Prof. K. Hess-Bellwald
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When $\mathscr{C} = \mathscr{P}(R)$ and $\mathscr{D} = \mathscr{P}(S)$, we have :

2.2.3 Proposition. Let $F : \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_S$ be a functor such that

- $F(M \oplus N) \cong FM \oplus FN$,
- $FR \in \mathscr{P}(S)$.

Then F restricts and corestricts to an exact functor

$$F : \mathscr{P}(R) \longrightarrow \mathscr{P}(S)$$

and therefore induces a homomorphism $K_0F : K_0R \longrightarrow K_0S$.

Proof. We need to show that $FP \in \mathscr{P}(S)$, $\forall P \in \mathscr{P}(R)$. We know that there exists $P' \in \mathscr{P}(R)$ such that $P \oplus P' \cong R^{\oplus n}$, for some $n \in \mathbb{N}$. Thus

$$\begin{aligned}
FP \oplus FP' &\cong F(P \oplus P') \\
&\cong F(R^{\oplus n}) \\
&\cong (FR)^{\oplus n} \\
&\in \mathscr{P}(S).
\end{aligned}$$

So $FP \in \mathscr{P}(S)$, and F does indeed give rise to $F : \mathscr{P}(R) \longrightarrow \mathscr{P}(S)$. It is exact since every exact sequence in $\mathscr{P}(R)$ splits, and that F preserves $\oplus$ up to isomorphism. ∎

Restruction of scalars

If $\phi : R \longrightarrow S$ is a homomomorphism (!) of rings, then it induces a functor

$$\begin{aligned}
\phi^* : \mathbf{Mod}_S &\longrightarrow \mathbf{Mod}_R \\
(M, \rho) &\longmapsto (M, \rho \circ (\mathrm{id}_M \times \phi)).
\end{aligned}$$

In particular, since S is a S-module, we can view it as a R-module. The functor ϕ^* does not change the underlying abeloan group, only the action. Moreover, if $f : (M, \rho) \longrightarrow (M', \rho')$ is a homomorphism pf S modules, then

$$\phi^* f : (M, \rho \circ (\mathrm{id}_M \times \phi)) \longrightarrow (M', \rho' \circ (\mathrm{id}_M \times \phi))$$

has the same underlying homomorphism of groups. Since f is a homomorphism of R module, we have

$$\begin{aligned} & f(sm) = sf(m), \qquad \forall s \in S, \forall m \in M \\ \Longrightarrow\ & f(rm) = f(\phi(r)m) \\ & \quad = \phi(r)f(m) \\ & \quad = rf(m), \end{aligned}$$

and $\phi^* f$ is a homomorphism of R-modules.

$$\begin{tikzcd} \mathbf{Mod}_S \arrow[rr, "\phi^*"] \arrow[dr, "\mathscr{U}"'] & & \mathbf{Mod}_R \arrow[dl, "\mathscr{U}"] \\ & \mathbf{Ab} & \end{tikzcd}.$$

Consequently, ϕ^* preserves exact sequences and is an exact functor. If $\phi^* S$ is finitely generated and projective an an R-module, then it restricts and corestricts to an exact functor $\phi^* : \mathscr{P}(S) \longrightarrow \mathscr{P}(R)$, and therefore induces an homomorphism

$$K_0\phi^* : K_0 S \longrightarrow K_0 R.$$

Not quite what we want to see K_0 as a functor $\mathbf{Ring} \longrightarrow \mathbf{Ab}$...

2.2.4 Example. Let $\phi : \mathbb{Z}/6\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ be the quotient homomorphism. Since $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, it comes that $\mathbb{Z}/2\mathbb{Z}$ is finitely generated and projective as a $\mathbb{Z}/6\mathbb{Z}$-module. So ϕ induces a homomorphism $K_0\phi^* : K_0(\mathbb{Z}/2\mathbb{Z}) \longrightarrow K_0(\mathbb{Z}/6\mathbb{Z})$.

Extension of scalars

Let $\phi : R \longrightarrow S$ be a ring homomorphism. Define functor

$$\begin{aligned} S \otimes_R - : {}_R\mathbf{Mod} &\longrightarrow {}_S\mathbf{Mod} \\ M &\longmapsto S \otimes_R M \\ f &\longmapsto S \otimes_R f = \mathrm{id}_S \otimes_R f, \end{aligned}$$

where S is implicitly considered as a (S, R)-bimodule.

- The fact that $S \otimes_R -$ preserves direct sum is a special case of additivity (exercise set 7),
- $S \otimes_R R \cong S$ is a finitely generated projective S-module.

So there exists a homomorphism

$$\begin{aligned} K_0(S \otimes_R -) : K_0 R &\longrightarrow K_0 S \\ [P] &\longmapsto [S \otimes_R P]. \end{aligned}$$

2.2.5 Notation. $K_0\phi = K_0(S \otimes_R -)$.

2.2.6 Theorem. With respect to this choice of $K_0\phi$,

$$K_0 : \mathbf{Ring} \longrightarrow \mathbf{Ab}$$

is a functor. It restricts and corestricts to a functor

$$K_0 : \mathbf{CRing} \longrightarrow \mathbf{CRing}.$$

Proof.
- We will show that $\forall R \xrightarrow{\phi} S \xrightarrow{\psi} T$, we have $K_0(\psi \circ \phi) = K_0\psi \circ K_0\phi$. Let $P \in \operatorname{Obj} \mathscr{P}(R)$,

$$\begin{aligned} K_0\psi \circ K_0\phi([P]) &= K_0\psi([S \otimes_R P]) \\ &= [T \otimes_S (S \otimes_R P)] \\ &= [T \otimes_R P] \\ &= K_0(\psi \circ \phi)([P]). \end{aligned}$$

- We will show that for all ring R, we have $K_0 \operatorname{id}_R = \operatorname{id}_{K_0 R}$. Remark that $\operatorname{id}_R^* R = R$, with the usual R-module structure. Therefore $R \otimes_R M \cong M$, with respect to id_R.

If R is a commutative ring, then $K_0 R$ as also commutative, with $[P][Q] = [P \otimes_R Q]$. Let $\phi : R \longrightarrow S$ is a homomorphism of ring. Then

$$\begin{aligned} K_0\phi([P])K_0\phi([Q]) &= [S \otimes_R P][S \otimes_R Q] \\ &= [(S \otimes_R P) \otimes_S (S \otimes_R Q)] \\ &= [((S \otimes_R P) \otimes_S S) \otimes_R Q] \\ &= [S \otimes_R (P \otimes_R Q)] \\ &= K_0\phi([P][Q]). \end{aligned}$$

□

Tensoring with bimodules

This is a generalisation of the extension of scalars. Let $M \in \operatorname{Obj} {}_S\mathbf{Mod}_R$. Then we have a functor

$$M \otimes_R - : {}_R\mathbf{Mod} \longrightarrow {}_S\mathbf{Mod}$$

that preserves direct sum. Is M is projective and finitely generated, then $M \otimes_R R \cong M \in \operatorname{Obj} \mathscr{P}(S)$. Whence $M \otimes_R -$ induces a homomorphism

$$K_0(M \otimes_R -) : K_0 R \longrightarrow K_0 S.$$

Central idempotents

This is a special case of the extension of scalars.

2.2.7 Lemma. Let R be a ring, and $e \in R$ a centran idempotent. Then eR is a ring with neutral element e.

Proof.
- $er + er' = e(r + r') \in eR$,
- $(er)(er') = erer' = e^2 rr' = e(rr') \in eR$.

So eR is closed under $+$ and $\cdot$, inherited by R. Moreover $(er)e = e(er) = e^2 r = er$, so e is the neutral element of eR. □

Let

$$\begin{aligned} \phi_e : R &\longrightarrow eR \\ r &\longmapsto er. \end{aligned}$$

This is a ring homomorphism since

$$\begin{aligned}\phi_e(rr') &= err' \\ &= e^2rr' \\ &= erer' \\ &= \phi_e(r)\phi_e(r').\end{aligned}$$

So we have an induces homomorphism

$$\begin{aligned}K_0\phi_e : K_0R &\longrightarrow K_0eR \\ [P] &\longmapsto [eR\otimes_R] = [eM] \qquad\qquad \text{(exercise !)}.\end{aligned}$$

To illustrate the utility of central idempotents, and as a method for computing K_0R :

2.2.8 Theorem. $K_0(R \times R') \cong K_0R \times K_0R'$.

Proof. Let $e = (1, 0)$ and $e' = (0, 1)$. It is obvious that they are central idempotents. Moreover

$$\begin{aligned}e(R \times R') &\cong R \\ e'(R \times R') &\cong R'.\end{aligned}$$

There exists homomorphism of ring

$$\begin{aligned}\phi_e : R \times R' &\longrightarrow e(R \times R') \\ \phi_{e'} : R \times R' &\longrightarrow e'(R \times R'),\end{aligned}$$

and therefore group homomorphism

$$\begin{aligned}K_0\phi_e : K_0(R \times R') &\longrightarrow K_0e(R \times R') \\ K_0\phi_{e'} : K_0(R \times R') &\longrightarrow K_0e'(R \times R'),\end{aligned}$$

from which we get

$$\begin{aligned}\alpha = (K_0\phi_e, K_0\phi_{e'}) : K_0(R \times R') &\longrightarrow K_0e(R \times R') \times K_0e'(R \times R') \\ [P] &\longmapsto ([eP], [e'P]).\end{aligned}$$

Claim : this is an isomorphism. Observe that there exists a split exact sequence of $(R\times R')$-modules

$$\begin{tikzcd} 0 \arrow[r] & e'(R \times R') \arrow[r, hook] & R \times R' \arrow[l, bend right, "\phi_{e'}"'] \arrow[r, "\phi_e"] & e(R \times R') \arrow[l, bend left, hook] \arrow[r] & 0 \end{tikzcd}.$$

So $R \times R' \cong e(R \times R') \oplus e'(R \times R')$. In particular, both $e(R \times R')$ and $e'(R \times R')$ are finitely generated projective $(R \times R')$-modules. Therefore ϕ_e and $\phi_{e'}$ induce homomorphism

$$\begin{aligned}K_0\phi_e^* : K_0e(R \times R') &\longrightarrow K_0(R \times R') \\ K_0\phi_{e'}^* : K_0e'(R \times R') &\longrightarrow K_0(R \times R').\end{aligned}$$

Claim : $K_0\phi_e^* \oplus K_0\phi_{e'}^*$ is the inverse of $(K_0\phi_e, K_0\phi_{e'})$. We have a homomorphism

$$\begin{aligned}\beta : (K_0e(R \times R')) \times K_0e'(R \times R')) &\longrightarrow K_0(R \times R') \\ ([M], [N]) &\longmapsto K_0\phi_e^*([M]) + K_0\phi_{e'}^*([N]).\end{aligned}$$

- We show that $\beta \circ \alpha = \mathrm{id}_{K_0(R\times R')}$. Let $[M] \in K_0(R \times R')$. We have

$$\begin{aligned}
\beta \circ \alpha([M]) &= \beta\left([e(R \times R') \otimes_{R\times R'} M], [e'(R \times R') \otimes_{R\times R'} M]\right) \\
&= [e(R \times R') \otimes_{R\times R'} M] + [e'(R \times R') \otimes_{R\times R'} M] \\
&= [(e(R \times R') \otimes_{R\times R'} M) \oplus (e'(R \times R') \otimes_{R\times R'} M)] \\
&= [(e(R \times R') \oplus e'(R \times R')) \otimes_{R\times R'} M] \\
&= [(R \times R') \otimes_{R\times R'} M] \\
&= [M].
\end{aligned}$$

- Conversly, il M is an $e(R \times R')$-module, then $eM = M$ since e is the neutral element of $e(R \times R')$. Similarly, if M' is an $e'(R \times R')$-module, then $e'M' = M'$. So

$$\begin{aligned}
\alpha \circ \beta([M], [M']) &= \alpha\left([\phi_e^* M] + [\phi_{e'}^* M']\right) \\
&= \left([e(R \times R') \otimes_{R\times R'} \phi_e^* M], [e'(R \times R') \otimes_{R\times R'} \phi_e M]\right) \\
&\quad + \left([e(R \times R') \otimes_{R\times R'} \phi_{e'}^* M'], [e'(R \times R') \otimes_{R\times R'} \phi_{e'} M']\right) \\
&= ([eM], [e'M]) + ([eM'], [e'M']) \\
&= ([M], 0) + (0, [M']) \\
&= ([M], [M']),
\end{aligned}$$

since $e'M = e'eM = 0M = 0$.

So

$$\begin{aligned}
K_0(R \times R') &\cong K_0 e(R \times R') \times K_0 e'(R \times R') \\
&\cong K_0 R \times K_0 R'.
\end{aligned}$$

□

2.3 Localization

Il this section, all rings are commutative. The idea here is to simplify a ring R by looking at it "one prime (ideal) at a time". We'll try to "invert" articicially as many elements as possible so that we get something close to a field. We know that $K_0\mathbb{K} \cong \mathbb{Z}$ as rongs when $\mathbb{K}$ is a field. We'll try to make R **local**, i.e. with only one maximal ideal. It turns out that $K_0 R$ is easy to calculate if that case. In exercise set 10, we'll see examples of local rings, computation of their K_0 and determine when localization produces local ring. In class, we'll see the theory of localization ant its importance for K-theory.

Localization is defined by a universal property :

2.3.1 Definition (Localization)**.** Let R be a commutative ring, and $S \subseteq R$ ba a subset. A **localization** of R away from S consists of a ring R' and a ring homomorphism $\phi : R \longrightarrow R'$ such that

- $\phi(s)$ is invertible in R', $\forall s \in S$,
- $\forall \psi : R \longrightarrow R''$ that satisfies the previous condition, $\exists! \widehat{\psi} : R' \longrightarrow R''$ such that the following

diagram commutes :

$$\begin{tikzcd}
R \arrow[r, "\phi"] \arrow[dr, "\psi"'] & R' \arrow[d, dashed, "\exists!\widehat{\psi}"] \\
& R''
\end{tikzcd}.$$

2.3.2 Remark. Sonce this is defined by a universal property, if the localization exists, it is unique.

2.3.3 Notation. We note $\iota_S = \phi$ and $S^{-1}R = R'$.

We want to invert elements of S. Wo elements of $S^{-1}R$ should look like $\frac{r}{s}$, as ine the construction of the quotient field of a domain.

2.3.4 Theorem. The localization always exists.

Proof. We give an explicit construction ! Let $X = \{x_s \mid s \in S\}$. Consider $R[X]$, the molynomial ring on X with coefficients in R (here, we need R to be commutative). Define

$$J_S = \langle sx_s - 1 \mid s \in S \rangle .$$

Let ι_S be the following composite :

$$\begin{tikzcd}
R \arrow[r, hook, "j"] & R[X] \arrow[r, two heads, "q"] & R[X]/J_S
\end{tikzcd}.$$

We have to check that is satisfies the required properties.

- If $s \in S$, than $\iota_S(s)$ is invertible in $R[X]/J_S$ since

$$\begin{aligned}
[x_s][s] &= [sx_s] \\
&= [1].
\end{aligned}$$

- If $\psi : R \longrightarrow R''$ is a ring homomorphism such that $\psi(s)$ is invertible in R', $\forall s \in S$, then consider

$$\begin{tikzcd}
R \arrow[r, hook, "j"'] \arrow[rr, bend left, "\iota_S"] \arrow[dr, "\psi"'] & R[X] \arrow[r, two heads, "q"'] \arrow[d, dashed, "\exists!\widetilde{\psi}"] & R[X]/J_S \arrow[dl, dashed, "\exists!\widehat{\psi}"] \\
& R'' &
\end{tikzcd} \quad \exists!\widehat{\psi} \iff \ker p = J_S \subseteq \ker \widetilde{\psi},$$

with $\widetilde{\psi}(x_s) = \psi(s)^{-1}$. Since

$$\begin{aligned}
\widetilde{\psi}(sx_s - 1) &= \widetilde{\psi}(s)\widetilde{\psi}(x_s) - \widetilde{\psi}(1) \\
&= \psi(s)\psi(s)^{-1} - 1 \\
&= 0,
\end{aligned}$$

we have that $\ker p = J_S \subseteq \ker \widetilde{\psi}$.

□

2.3.5 Remark. If $0 \in S$, then in $S^{-1}R$ we have $[x_0] = [0]^{-1}$. So

$$\begin{aligned}[0] &= [0x_0] \\ &= [0][x_0] \\ &= [1] \\ &\implies S^{-1}R = \{0\}.\end{aligned}$$

What happens if S contains zero divisors ?

2.3.6 Notation. For any $S \subseteq R$, let $\overline{S}$ be the multiplicative closure of S, i.e.

$$\overline{S} = \{s_1 \cdots s_n \mid n \in \mathbb{N}, s_i \in S, \forall 1 \leq i \leq n\}.$$

2.3.7 Proposition. Let R be a commutative ring and $S \subseteq R$.

1. We have
$$\ker \iota_S = \left\{r \in R \mid \exists \overline{s} \in \overline{S} \text{ such that } r\overline{s} = 0\right\}.$$
In particular, if R doesn't contain zero divisor, then ι_S is an embedding.

2. (A more manageable description of $S^{-1}R$) : $\forall \gamma \in S^{-1}R$, $\exists r \in R$, $\exists \overline{s} \in \overline{S}$ (not necessarily unique) such that
$$\gamma = \iota_S(\overline{s})^{-1}\iota_S(r).$$

Proof. 1.

$\supseteq$: We have

$$\begin{aligned}0 &= \iota_S(0) \\ &= \iota_S(\overline{s}r) \\ &= \iota_S(\overline{s})\iota_S(r).\end{aligned}$$

So

$$\begin{aligned}0 &= \iota_S(\overline{s})^{-1}0 \\ &= \iota_S(\overline{s})^{-1}\iota_S(\overline{s})\iota_S(r) \\ &= \iota_S(r) \\ &\implies r \in \ker \iota_S.\end{aligned}$$

$\subseteq$: Let $r \in \ker \iota_S$.

$$\begin{aligned}r \in \ker \iota_S &\implies \iota_S(r) = 0 \\ &\implies r \in J_S \\ &\implies \exists s_1, \ldots, s_n \in S, \exists p_1, \ldots, p_n \in R[X] \text{ such that } r = \sum_{i=1}^{n} p_i \cdot (s_1 x_s - 1).\end{aligned}$$

So in that case, it is enough to consider $S' = \{s_1, \ldots, s_n\} \subseteq S$. Since $J_{S'} \subseteq J_S$, there exists a ring homomorphism (quotient map)

$$\begin{tikzcd} & R \arrow[dl, "\iota_{S'}"'] \arrow[dr, "\iota_S"] & \\ (S')^{-1}R \arrow[rr] & & S^{-1}R \end{tikzcd}.$$

Moreover, $r \in \ker \iota_{S'}$ since $r \in J_{S'}$. Let $\overline{s} = \prod_{i=1}^{n} s_i$. In the exercise set, we have shown that $(S')^{-1}R \cong \{\overline{s}\}^{-1}R$. Since $r \in \ker \iota_{S'} = \ker \iota_{\{\overline{s}\}}$, we have $r \in \langle \overline{s}x_{\overline{s}} - 1\rangle$, i.e. $\exists p \in R[x_{\overline{s}}]$ such that

$$r = p(x_{\overline{s}}) \cdot (\overline{s}x_{\overline{s}} - 1).$$

Write $p(x_{\overline{s}}) = \sum_{i=0}^{n} a_i x_{\overline{s}}^i$, with $a_i \in R$. Then

$$\begin{aligned} r &= p(x_{\overline{s}}) \cdot (\overline{s}x_{\overline{s}} - 1) \\ &= -a_0 + (a_0\overline{s} - a_1)x_{\overline{s}} + (a_1\overline{s} - a_2)x_{\overline{s}}^2 + \cdots + (a_{n-1}\overline{s} - a_n)x_{\overline{s}}^n + a_n\overline{s}x_{\overline{s}}^{n+1}. \end{aligned}$$

Thus, $r = -a_0$, $a_i\overline{s} - a_{i+1} = 0$, $\forall 0 \leq i \leq n-1$, and $a_x\overline{s} = 0$. So $a_{i+1} = a_i\overline{s}$, and $r\overline{s}^{n+1} = a_n\overline{s} = 0$. Since $\overline{s}^{n+1} = 0$, we have the inclusion.

2. Let $R' = \{\iota_S(\overline{s})^{-1}\iota_S(r) \mid \overline{s} \in \overline{S}, r \in R\} \subseteq S^{-1}R$. We want to show that $R' = S^{-1}R$. First, note that R' is a subring of $S^{-1}R$. Indeed :

   - $(\iota_S(\overline{s})^{-1}\iota_S(r))(\iota_S(\overline{s}')^{-1}\iota_S(r')) = \iota_S(\overline{s}\overline{s}')^{-1}\iota_S(rr')$, and $\overline{s}\overline{s}' \in \overline{S}$,
   - think of it as fractions :

   $$\begin{aligned} (\iota_S(\overline{s})^{-1}\iota_S(r)) + (\iota_S(\overline{s}')^{-1}\iota_S(r')) &= \iota_S(\overline{s}\overline{s}')^{-1}(\iota_S(\overline{s}'r) + \iota_S(\overline{s}r')) \\ &= \iota_S(\overline{s}\overline{s}')^{-1}\iota_S(\overline{s}r' + \overline{s}r'), \end{aligned}$$

   and $\overline{s}r' + \overline{s}r' \in R$.

   So R' is indeed a subring. Consider

   $$\begin{tikzcd} R \arrow[r, "\iota_S"] \arrow[dr, "\iota'"] \arrow[drr, bend right, "\iota_S"'] & S^{-1}R \arrow[d, dashed, "\exists!\widehat{\iota'}"] \arrow[dr, "\mathrm{id}_{S^{-1}R}"] & \\ & R' \arrow[r, hook, "j"] & S^{-1}R \end{tikzcd}.$$

   We have $j \circ \widehat{\iota'} = \mathrm{id}_{S^{-1}R}$, so j is a surjection, and $R' \cong S^{-1}R$.

$\square$

Here is a sketch of a "fraction" approach to localization. It generalizes more easily to non commutative rings. Define

$$S^{-1}R = (R \times S)/\sim,$$

where

$$(r, s) \sim (r', s') \iff \exists t \in S \text{ such that } t(rs' - r's) = 0.$$

Let $\frac{r}{s} = [(r,s)]$. We have

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'},$$
$$\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'},$$

$$\begin{aligned} \iota_S : R &\longrightarrow S^{-1}R \\ r &\longmapsto \frac{rs}{s} \qquad \text{for some } s \in S. \end{aligned}$$

Then, $\iota_S(s)$ is invertible, with inverse $\frac{s}{s^2}$. We have the universal property :

$$\begin{tikzcd} R \arrow[r, "\iota_S"] \arrow[dr, "\psi"'] & S^{-1}R \arrow[d, dashed, "\exists!\widehat{\psi}"] \\ & R' \end{tikzcd},$$

with $\widehat{\psi}(\frac{r}{s}) = \psi(s)^{-1}\psi(r)$.

2.3.8 Examples.

1. If $\mathfrak{p}$ is a prime ideal of R, then we note $R_{\mathfrak{p}} = (R \setminus \mathfrak{p})^{-1}R$ (R localized at $\mathfrak{p}$). If $R = \mathbb{Z}$, $p \in \mathbb{Z}$ is a prime number, then $p\mathbb{Z}$ is a prime ideal. We note $\mathbb{Z}_{(p)} = \mathbb{Z}_{p\mathbb{Z}}$ (the ring of integers localized at p).
2. Choose $s \in R$. Then $R\left[\frac{1}{s}\right] = \{s\}^{-1}R = \{s, s^2, \ldots\}^{-1}R$ (R localized away from s). For example, if $R = \mathbb{Z}$ and if $p \in \mathbb{Z}$ is a prime number, then $\mathbb{Z}\left[\frac{1}{p}\right]$ is the localization of $\mathbb{Z}$ away from p.

What can we say about $K_0\iota_S : K_0R \longrightarrow K_0S^{-1}R$? We'll compute $\ker K_0\iota_S$. For that, we need to understand $K_0\iota_S([P]) = [S^{-1}R \otimes_R P]$, at least for $P \in \operatorname{Obj} \mathscr{P}(R)$.

2.3.9 Notation. If M is a R-module, then $S^{-1}M = S^{-1}R \otimes_R M \in \operatorname{Obj}{}_{S^{-1}R}\mathbf{Mod}$.

2.3.10 Remark. $S^{-1}M$ also satisfies a universal property, like that satisfied by $S^{-1}R$ (exercise !).

2.3.11 Lemma. Let $M \in \operatorname{Obj}{}_R\mathbf{Mod}$. Consider the homomorphism of R-modules given by

$$\begin{aligned} \iota_S \otimes_R \mathrm{id}_M : R \otimes_R M \cong M &\longrightarrow S^{-1}R \otimes_R M = S^{-1}M \\ m &\longmapsto 1 \otimes m. \end{aligned}$$

Then

$$\ker(\iota_S \otimes_R \mathrm{id}_M) = \left\{ m \in M \mid \exists \overline{s} \in \overline{S} \text{ such that } \overline{s}m = 0 \right\}.$$

Proof. Exercise. This is slightly more technical than the calculation of $\ker \iota_S$... $\square$

To better understand what the elements of $S^{-1}M$ are

2.3.12 Lemma.

1. Wa have

$$S^{-1}M = \left\{ \iota_S(\overline{s})^{-1} \otimes m \mid \overline{s} \in \overline{S}, m \in M \right\},$$

where $(\iota_S(\overline{s}_1)^{-1} \otimes m_1) + (\iota_S(\overline{s}_2)^{-1} \otimes m_2) = \iota_S(\overline{s}_1, \overline{s}_2)^{-1} \otimes (\overline{s}_2 m_1 + \overline{s}_1 m_2)$. Think of addition of fractions.

2. $\iota_S(\overline{s}_1)^{-1} \otimes m_1 = \iota_S(\overline{s}_2)^{-1} \otimes m_2$ in $S^{-1}M$ iff $\exists \overline{s} \in \overline{S}$ such that $\overline{s}(\overline{s}_2 m_1 - \overline{s}_1 m_2) = 0$ in M.

Proof. 1. We have $S^{-1}M = S^{-1}R \otimes_R M$. Its typical elements are of the form

$$\begin{aligned}\sum_{i=1}^{n} (\iota_S(\overline{s}_i)^{-1}\iota_S(r_i)) \otimes m_i &= \sum_{i=1}^{n} \iota_S(\overline{s}_i)^{-1} \otimes (\iota_S(r_i)m_i) \\ &= \iota_S(\overline{s}_1 \cdots \overline{s}_n)^{-1} \otimes \left(\sum_{i=1}^{n} \overline{s}_1 \cdots \widehat{\overline{s}_i} \cdots \overline{s}_n \cdot r_i m_i \right).\end{aligned}$$

2. We have :

$$\begin{aligned}\iota_S(\overline{s}_1)^{-1} \otimes m_1 = \iota_S(\overline{s}_2)^{-1} \otimes m_2 \text{ in } S^{-1}M &\iff \iota_S(\overline{s}_1)^{-1} \otimes m_1 - \iota_S(\overline{s}_2)^{-1} \otimes m_2 = 0 \text{ in } S^{-1}M \\ &\iff \iota_S(\overline{s}_1\overline{s}_2)^{-1} \otimes (\overline{s}_2 m_1 - \overline{s}_1 m_2) = 0 \text{ in } S^{-1}M \\ &\iff \underbrace{1 \otimes (\overline{s}_2 m_1 - \overline{s}_1 m_2)}_{=(\iota_S \otimes_R \mathrm{id}_M)(\overline{s}_2 m_1 - \overline{s}_1 m_2)} = 0 \text{ in } S^{-1}M \qquad \text{(multiplicat} \\ &\iff \exists \overline{s} \in \overline{S} \text{ such that } \overline{s}(\overline{s}_2 m_1 - \overline{s}_1 m_2) = 0 \text{ in } M \qquad \text{by lemma 2}\end{aligned}$$

□

2.3.13 Properties. 1. The functor $S^{-1}R \otimes_R - : {}_R\mathbf{Mod} \longrightarrow {}_{S^{-1}R}\mathbf{Mod}$ is exact, i.e. if $0 \to L \to M \to N \to 0$ is exact in ${}_R\mathbf{Mod}$, then $0 \to S^{-1}L \to S^{-1}M \to S^{-1}N \to 0$ is exact in ${}_{S^{-1}R}\mathbf{Mod}$. We say that $S^{-1}R$ is a **flat** R-module.

2. If $M \in \mathrm{Obj}\,\mathscr{M}(R)$, then $S^{-1}M = 0$ (we say that M is an S-toreion module) iff $\exists \overline{s} \in \overline{S}$ such that $\overline{s}M = \{0\}$.

3. (Realizability of $S^{-1}R$-modules, or surjectivity of $S^{-1}R \otimes_R -$ up to isomorphism) For all $N \in \mathrm{Obj}\,\mathscr{M}(S^{-1}R)$, $\exists M \in \mathrm{Obj}\,\mathscr{M}(R)$ such that $S^{-1}M \cong N$.

4. (Realizability of isomorphisms between $S^{-1}R$-modules, or uniqueness of realizability of $S^{-1}R$-modules) Let $M, N \in \mathrm{Obj}\,\mathscr{M}(R)$ such that $\forall s \in S$, s acts injectively on M and N. Then $S^{-1}M \cong S^{-1}N$ iff $\exists N' \leq N$ such that
   - $N' \cong M$,
   - N/N' is an S-torsion module, i.e. $S^{-1}(N/N') = 0$.

   We say that M and N are isomorphic up to S-torsion.

Proof. 1. In exercise set, we showed that if $0 \to L \overset{j}{\to} M \overset{p}{\to} N \to 0$ is exact in ${}_R\mathbf{Mod}$, then $S^{-1}L \overset{S^{-1}j}{\to} S^{-1}M \overset{S^{-1}p}{S}{}^{-1} N \to 0$ is exact in ${}_{S^{-1}R}\mathbf{Mod}$ (property of the tensor product $\otimes_R$). We have to show that $S^{-1}j = \mathrm{id}_{S^{-1}R} \otimes_R j$ is injective. Suppose that $\iota_S(\overline{s})^{-1} \otimes l \in \ker S^{-1}j$. Then $\iota_S(\overline{s})^{-1} \otimes j(l) = 0 = \iota_S(\overline{s})^{-1} \otimes 0$. So, by lemma 2.3.12, we know that $\exists \overline{s}' \in \overline{S}$ such that $\overline{s}'\overline{s}j(l) = 0$. But

$$\begin{aligned} 0 &= \overline{s}'\overline{s}j(l) \\ &= j(\overline{s}'\overline{s}l) \\ \implies \overline{s}'\overline{s}l &= 0,\end{aligned}$$

since j is injective. By lemma 2.3.11, $l \in \ker(\iota_S \otimes_R \mathrm{id}_L)$, i.e. $1 \otimes l = 0$ and so $\iota_S(\overline{s})^{-1} \otimes l = 0$. So $\ker S^{-1}j = \{0\}$, and $S^{-1}j$ is injective.

2.

$\Longrightarrow$: If $\overline{s}M = \{0\}$, then $\ker(\iota_S \otimes_R \mathrm{id}_M) = M$, by lemma 2.3.11, and so $S^{-1}M = 0$. More explicitely, if $\overline{s}m = 0$, then

$$\begin{aligned} 1 \otimes m &= (\iota_S(\overline{s})^{-1}\iota_S(\overline{s})) \otimes m \\ &= \iota_S(\overline{s})^{-1} \otimes (\iota_S(\overline{s})m) \\ &= 0 \\ \Longrightarrow \iota_S(\overline{s}')^{-1} \otimes m &= \iota_S(\overline{s}')(1 \otimes m) \\ &= 0. \end{aligned}$$

Remark that we don't actually need M to be finitely generated.

$\Longleftarrow$: Suppose $M \in \mathrm{Obj}\, calMM(R)$ and that $S^{-1}M = 0$, and so that $1^o xm = 0$ in $S^{-1}M$, $\forall m \in M$, i.e. $M = \ker(\iota_S \otimes_R \mathrm{id}_M)$. Since M is finitely generated, $\exists x_1, \dots, x_n \in M$ sich that $M = \sum_{i=1}^n Rx_i$. In particular, $x_i \in \ker(\iota_S \otimes_R \mathrm{id}_M)$. By lemma 2.3.11, $\exists \overline{s}_i \in \overline{S}$ such that $\overline{s}_i x_i = 0$. Let $\overline{s} = \overline{s}_1 \cdots \overline{s}_n$. Then $\overline{s}x_i = 0$ since R is commutative, and so $\overline{s}M = \{0\}$.

3. Let $N \in \mathscr{M}(S^{-1}R)$. Since N is finitely generated, $\exists x_1, \dots, x_n \in N$ such that $N = \sum_{i=1}^n S^{-1}Rx_i$. Recall that since $\iota_S : R \longrightarrow S^{-1}R$ is a homomorphism of ring, there exists a functor $\iota_S^* : {}_R\mathbf{Mod} \longrightarrow {}_{S^{-1}R}\mathbf{Mod}$. Ler $M = \sum_{i=1}^n Rx_i$ be a submodule ob $\iota_S^* N$. Then

$$\begin{aligned} S^{-1}M &= S^{-1}R \otimes_R M \\ &= S^{-1}R \otimes_R \sum_{i=1}^n Rx_i \\ &= \left\{ \iota_S(\overline{s})^{-1} \otimes \sum_{i=1}^n r_i x_i \mid \overline{s} \in \overline{S}, r_i \in R \right\} \\ &= \left\{ \sum_{i=1}^n (\iota_S(\overline{s})^{-1} r_i \otimes x_i) \mid \overline{s} \in \overline{S}, r_i \in R \right\}. \end{aligned}$$

Then $S^{-1}N = \left\{ \sum_{i=1}^n (\iota_S(\overline{s}_i)^{-1} r_i \otimes x_i) \mid \overline{s}_i \in \overline{S}, r_i \in R \right\}$. By using fraction formula for addition in $S^{-1}R$, we can convert any element in $S^{-1}N$ into one of the form in $S^{-1}R \otimes_R M$ (micro-exercise). So $S^{-1}R \otimes_R M \cong N$.

4.

$\Longrightarrow$: Let $x_1, \dots, x_n$ be generators of M, i.e. $M = \sum_{i=1}^n Rx_i$. Since $sm \neq 0$, $\forall m \in M \setminus \{0\}$, we have that $\ker(\iota_S \otimes_R \mathrm{id}_M) = \{0\}$. We have

$$\begin{tikzcd} M = \sum_{i=1}^n \arrow[r, "\alpha", "\text{restr. / corestr. of } \alpha"'] \arrow[d, hook, "\iota_S \otimes_R \mathrm{id}_M"'] & \sum_{i=1}^n R\alpha(x_i) \arrow[d, hook] \\ S^{-1}M \cong \sum_{i=1}^n S^{-1}Rx_i \arrow[r, "\alpha \quad \cong", "x_i \longmapsto \alpha(x_i)"'] & S^{-1}N \end{tikzcd}.$$

We have that $\alpha(x_i) \in S^{-1}N$ implies that $\exists \overline{s}_i \in \overline{S}$ and $\exists y_i \in N$ such that $\alpha(x_i) = \iota_S(\overline{s}_i)^{-1} \otimes y_i$. Let $\overline{s} = \overline{s}_1 \cdots \overline{s}_n$. Then $\alpha(x_i) = \iota_S(\overline{s})^{-1} \otimes (\overline{s}_1 \cdots \widehat{\overline{s}_i} \cdots \overline{s}_n y_i)$. Now, there exist an isomorphism

$$\begin{aligned}\overline{s}\cdot - : \sum_{i=1}^{n} R\alpha(x_i) \le S^{-1}N &\longrightarrow \sum_{i=1}^{n} Ry_i = N' \le N\\ \alpha(x_i) &\longmapsto \overline{s}\alpha(x_i) = y_i.\end{aligned}$$

So $M \cong \sum_{i=1}^{n} R\alpha(x_i) \cong N'$. Finally, consider the exact sequence $0 \to N' \hookrightarrow N \to N/N' \to 0$ of R-modules, then apply $S^{-1}R \otimes_R -$ to get $0 \to S^{-1}N' \to S^{-1}N \to S^{-1}(N/N') \to 0$. But

$$\begin{aligned}S^{-1}N' &= S^{-1}R \otimes_R \sum_{i=1}^{n} Ry_i\\ &\cong \sum_{i=1}^{n} S^{-1}Ry_i\\ &\cong S^{-1}N,\end{aligned}$$

and so $S^{-1}(N/N') = 0$.

$\Longleftarrow$: Suppose that we have $N' \le N$ such that $N' \cong M$ and $S^{-1}(N/N') = 0$. Consider the exact sequence

$$0 \longrightarrow N' \hookrightarrow N \longrightarrow N/N' \longrightarrow 0$$

$$\downarrow S^{-1}R \otimes_R -$$

$$0 \longrightarrow S^{-1}N' \longrightarrow S^{-1}N \longrightarrow \underbrace{S^{-1}(N/N')}_{=0} \longrightarrow 0$$

and so $S^{-1}N \cong S^{-1}N' \cong S^{-1}M$.

□

We are interested in these properties of $S^{-1}R \otimes_R -$ as they allow us to prove

2.3.14 Theorem (Localization)**.** Let R be a commutative ring, $S \subset R$ be a subset that does not contains 0 nor zero divisors. Let $\mathscr{T}^2_{R,S}$ be the full subcategory of $\mathscr{M}(R)$ with objects

$$\operatorname{Obj} \mathscr{T}^2_{R,S} = \left\{ M \in \mathscr{M}(R) \mid S^{-1}M = 0 \text{ and } \operatorname{pd} M \le 2 \right\}.$$

Then ther exists an exact sequence in **Ab** :

$$K_0\mathscr{T}^2_{R,S} \xrightarrow{\widehat{\chi}} K_0R \xrightarrow{K_0\iota_S} K_0S^{-1}R \quad .$$

Proof. Define

$$\begin{aligned}\chi : \operatorname{Iso} \mathscr{T}^2_{R,S} &\longrightarrow K_0R\\ [M] &\longmapsto [P_0] - [P_1],\end{aligned}$$

where $0 \to P_1 \to P_0 \to M \to 0$ is a projective resolution of M. It is well defined and a generalized rank by Schanuel's lemma. We therefore have an induced homomorphism :

$$\begin{aligned} \widehat{\chi} : K_0 \mathscr{T}^2_{R,S} &\longrightarrow K_0 R \\ [M] &\longmapsto [P_0] - [P_1], \end{aligned}$$

$\operatorname{im} \widehat{\chi} \subseteq \ker K_0 \iota_S$: Let $0 \to P_1 \to P_0 \to M \to 0$ be a projective resolution of M. We apply the exact functor $S^{-1}R \otimes_R -$ to get another exact sequence

$$0 \longrightarrow S^{-1}P_1 \xrightarrow{\cong} S^{-1}R_0 \longrightarrow \underbrace{S^{-1}M}_{=0} \longrightarrow 0 \quad .$$

So $K_0\iota_S \circ \widehat{\chi}([M]) = 0$.

$\ker K_0\iota_S \subseteq \operatorname{im} \widehat{\chi}$: If $K_0\iota_S([P] - [Q]) = 0$ for $[P], [Q] \in K_0R$, then $[S^{-1}P] - [S^{-1}Q] = 0$, i.e. $[S^{-1}P] = [S^{-1}Q]$ in $K_0S^{-1}R$, i.e. $S^{-1}P \cong_S S^{-1}Q$, i.e. $\exists n \in \mathbb{N}$ such that $S^{-1}(P \oplus R^{\oplus n}) \cong S^{-1}P \oplus (S^{-1}R)^{\oplus n} \cong S^{-1}Q \oplus (S^{-1}R)^{\oplus n} \cong S^{-1}(Q \oplus R^{\oplus n})$. Observe that since S contains neither 0 nor zero divisors, it act injectively on any free module $R^{\oplus n}$, since $sr \neq 0$, $\forall s \in S$, $\forall r \in R$. Consequently, since any projective R-module is a summand of a free module, S also acts injectively on any projective R-module. So S acts injectively on $P \oplus R^{\oplus n}$ and $Q \oplus R^{\oplus n}$. Since both modules are finitely generated, we can apply a previous proposition. We get that there exists $N \leq Q \oplus R^{\oplus n}$ such that

- $N \cong P \oplus R^{\oplus n}$,
- $(Q \oplus R^{\oplus n})/N$ is a S-torsion module.

Consider the following exact sequence :

$$0 \longrightarrow \underbrace{N}_{\in \mathrm{Obj}\, \mathscr{P}(R)} \longrightarrow \underbrace{Q \oplus R^{\oplus n}}_{\in \mathrm{Obj}\, \mathscr{P}(R)} \longrightarrow \underbrace{(Q \oplus R^{\oplus n})/N}_{S\text{-torsion}} \longrightarrow 0 \quad .$$

We have that $(Q \oplus R^{\oplus n})/N \in \mathrm{Obj}\, \mathscr{T}^2_{R,S}$. Moreover :

$$\begin{aligned} \widehat{\chi}([(Q \oplus R^{\oplus n})/N]) &= [Q \oplus R^{\oplus n}] - [N] \\ &= [Q \oplus R^{\oplus n}] - [P \oplus R^{\oplus n}] \\ &= [Q] - [P]. \end{aligned}$$

□

Chapter 3

K_1 and classification of invertible matrices

The idea is to study an abstract version of the notion of determinant

$$\det : \mathrm{GL}_n(R) \longrightarrow R^*.$$

Our plan is :

1. see a matrix-theoretic definition of K_1, good for establishing properties, but bad for calculations,

2. determine the universal property of matrix-theoretic K_1, and get some computational tools,

3. see a Grothendieck-type description, clarify the relationship with K_0, and start to see how K-theory is a sort of homology theory for rings.

3.1 Matrix-theoretical approach to K_1

3.1.1 Notations.

- Let $\mathrm{Mat}_n(R)$ be the ring of n by n matrices with coefficients in R.
- Let $\mathrm{GL}_n(R) = \mathrm{Mat}_n(R)^*$ be the group of invertible matrices.
- $\forall 1 \leq k, l \leq n$, $k \neq l$, define $E_{k,l} \in \mathrm{Mat}_n(R)$ to be the matrix specified by

$$(E_{k,l})_{i,j} = \begin{cases} 1 & \text{if } i = k, j = l \\ 0 & \text{otherwise.} \end{cases}$$

- $\forall 1 \leq k, l \leq n$, $k \neq l$, $\forall r \in R$, define $\tau_{k,l}(r) = I_n + rE_{k,l}$.
- Let $E_n(R) = \langle \tau_{k,l}(r) \mid 1 \leq k, l \leq n, k \neq l, r \in R \rangle$.

3.1.2 Lemma. If $n \geq 3$, then $[E_n(R), E_n(R)] = E_n(R)$. Consequently, $E_n(R) \leq [\mathrm{GL}_n(R), \mathrm{GL}_n(R)]$.

Proof. Exercise set 12, exercise 1. □

3.1.3 Remark. For all $n \geq 1$, we have an injective homomorphism :

$$\begin{aligned} \mathrm{GL}_n(R) &\hookrightarrow \mathrm{GL}_{n+1}(R) \\ A &\longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

We then have a sequence of injective homomorphisms :

$$R^* = \mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}_2(R) \hookrightarrow \mathrm{GL}_3(R) \hookrightarrow \cdots$$

Define $\mathrm{GL}(R)$ to be the colimit of this diagram :

$$\mathrm{GL}(R) = \left\{ \begin{pmatrix} A & 0 \\ 0 & I_\infty \end{pmatrix} \mid \exists n \in \mathbb{N} \text{ such that } A \in \mathrm{GL}_n(R) \right\},$$

the ∞-dimentional general linear group. Note that the inclusion $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$ restricts and corestricts to an inclusion $E_n(R) \hookrightarrow E_{n+1}(R)$. We define $E(R) \leq \mathrm{GL}(R)$ in the same way.

3.1.4 Lemma (Whitehead)**.** We have $[\mathrm{GL}_n(R), \mathrm{GL}_n(R)] \leq E_{2n}(R)$, both seen as subgroups of $\mathrm{GL}(R)$.

Proof. Exercise set 12. $\square$

3.1.5 Corollary. $[\mathrm{GL}(R), \mathrm{GL}(R)] = E(R)$.

Proof.
- We show that $E(R) \leq [\mathrm{GL}(R), \mathrm{GL}(R)]$. Let $A \in E_n(R)$, seen as a $\mathrm{GL}(R)$ matrix $\begin{pmatrix} A & 0 \\ 0 & I_\infty \end{pmatrix}$. Since $A \in [\mathrm{GL}_n(R), \mathrm{GL}_n(R)]$ by earlyer lemma, and, seen as a subgroup of $\mathrm{GL}(R)$, we conclude that
$$\begin{pmatrix} A & 0 \\ 0 & I_\infty \end{pmatrix} \in [\mathrm{GL}(R), \mathrm{GL}(R)].$$
- We show that $[\mathrm{GL}(R), \mathrm{GL}(R)] \leq E(R)$. Let $A \in [\mathrm{GL}_n(R), \mathrm{GL}_n(R)]$, seen as a $\mathrm{GL}(R)$ matrix : $\begin{pmatrix} A & 0 \\ 0 & I_\infty \end{pmatrix}$. By the Whitehead lemma, $\begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix} \in E_{2n}(R)$, whence
$$\begin{pmatrix} A & 0 \\ 0 & I_\infty \end{pmatrix} \in E(R).$$

$\square$

3.1.6 Definition (Bass–Whitehead group)**.** Let R be a ring. The **Baas–Whitehead group** of R is defined by

$$K_1 R = \mathrm{GL}(R)/E(R).$$

It is the abelianization of $\mathrm{GL}(R)$.

3.1.7 Proposition. K_1 extends to a functor $K_1 : \mathbf{Ring} \longrightarrow \mathbf{Ab}$.

Proof. Let $\phi : R \longrightarrow S$ be a ring homomorphism. Consider the following diagram of groups :

$$\begin{tikzcd} E(R) \arrow[r, "E(\phi)"] \arrow[d, hook] & E(S) \arrow[d, hook] \\ \mathrm{GL}(R) \arrow[r, "\mathrm{GL}(\phi)"] \arrow[d, two heads] & \mathrm{GL}(S) \arrow[d, two heads] \\ K_1R \arrow[r, dashed, "\exists! K_1\phi"] & K_1S \end{tikzcd}.$$

□

3.1.8 Properties. 1. $K_1R \cong K_1(R^{\mathrm{op}})$.

2. $K_1 \operatorname{Mat}_n(R) \cong K_1R$.
3. $K_1(R \times R') \cong K_1R \cong K_1R'$.

Proof. Exercise set 12. □

First hint of the relation between K_0 and K_1 :

- In K_0, we know that $[P] = [Q]$ iff $\exists n \in \mathbb{N}$ such that $P \oplus R^{\oplus n} \cong Q \oplus R^{\oplus n}$ (existanceof basis and dimension of modules).
- In K_1R, recall that $A \in GL_n(R)$ implies that the rows of A are a basis of a free module. Moreover, $[A] = [B]$ in K_1R iff $\exists E \in E(R)$ such thah $A = EB$. The bases determined by A and B are related by row operations. So K_1R tells us about uniqueness of bases up to row operations.

We'll make all this more precise...

3.2 The universal property of K_1R

3.2.1 Definition (Generalized determinant)**.** Let R be a ring. A **generalized determinant** on R is a sequence of maps $\{\delta_n : \mathrm{GL}_n(R) \longrightarrow G\}_{n\in\mathbb{N}}$, where G is an abelian group, and such that

1. $\delta_n(AB) = \delta_n(A)\delta_n(B)$, $\forall A, B \in \mathrm{GL}_n(R)$, $\forall n \in \mathbb{N}$,
2. $\delta_n(\tau_{k,l}(r)) = 1$, $\forall 1 \leq k, l \leq n$, $k \neq l$, $\forall r \in R$, $\forall n \in \mathbb{N}$,
3. the following diagram commutes :

$$\begin{tikzcd} \mathrm{GL}_n(\mathbb{F}) \arrow[rr, hook] \arrow[dr, "\delta_n"'] & & \mathrm{GL}_{n+1}(\mathbb{F}) \arrow[dl, "\delta_{n+1}"] \\ & G & \end{tikzcd}.$$

3.2.2 Examples. 1. Let R be a commutative ring. Define $\det_n : \mathrm{GL}_n(R) \longrightarrow R^*$ to be the usual determinant. Then $\det = \{\det_n\}_{n\in\mathbb{N}}$ is a generalized determinant.

2. (Stabilization) Let $s_n : \mathrm{GL}_n(R) \longrightarrow K_1R$ denote the composite

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, hook] & \mathrm{GL}(R) \arrow[r, two heads] & K_1R \end{tikzcd}.$$

3.2.3 Theorem. Stabilization is a universal generaized determinant, i.e. every other generalized determinant factors uniquely through s :

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, "s_n"] \arrow[dr, "\delta_n"'] & K_1R \arrow[d, dashed, "\exists!\widehat{\delta}"] \\ & G \end{tikzcd}.$$

Proof. The family of homomorphisms $\delta_n : \mathrm{GL}_n(R) \longrightarrow G$ induces a homomorphism

$$\begin{aligned} \widetilde{\delta} : \mathrm{GL}(R) &\longrightarrow G \\ \begin{pmatrix} A & 0 \\ 0 & I_\infty \end{pmatrix} &\longmapsto \delta_k(A), \end{aligned}$$

where $A \in \mathrm{GL}_k(R)$. It is well defined by a property of a generalized determinant. Note that $\widetilde{\delta}$ is defined precisely so that

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, "\delta_n"] \arrow[dr] & G \arrow[d, "\widetilde{\delta}"] \\ & \mathrm{GL}(R) \end{tikzcd}.$$

On the other hand, $\{\delta_n\}_{n\in\mathbb{N}^*}$ is a generalized determinant, so $\delta_n(\tau_{i,j}(r)) = 1$, $\forall 1 \leq i,j \leq n$, $\forall r \in R$. Consequently, $E_n(R) \subseteq \ker \delta_n$ and so $E(R) \subseteq \ker \widetilde{\delta}$. Thus there exists a induces homomorphism

$$\begin{tikzcd} \mathrm{GL}(R) \arrow[r, "\widetilde{\delta}"] \arrow[dr, two heads] & G \arrow[d, dashed, "\exists!\widehat{\delta}"] \\ & K_1R = \mathrm{GL}(R)/E(R) \end{tikzcd}.$$

Observe that

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, hook] \arrow[rr, bend left, "s_n"] \arrow[dr, "\delta_n"'] & \mathrm{GL}(R) \arrow[r, two heads] \arrow[d, dashed, "\exists!\widetilde{\delta}"] & K_1R \arrow[dl, dashed, "\exists!\widehat{\delta}"] \\ & G & \end{tikzcd}.$$

□

3.3 A Grothendieck type approach to K_1

Observe that invertible matricies correspond to automorphisms of free R-modules. So do a Grothendieck type construction, taking automorphisms of modules into account.

3.3.1 Definition (Bass K_1 group)**.** Let R be a ring. Let $\mathscr{C}$ be a subcategory of ${}_R\mathbf{Mod}$ with a set of isomorphism classes of objects. The **Bass K_1 group** of $\mathscr{C}$ is

$$K_1\mathscr{C} = (F_{\mathbf{Ab}}\,\mathrm{Iso}\{(P,\alpha) \mid P \in \mathrm{Obj}\,\mathscr{C}, \alpha \in \mathrm{Aut}(P)\})/G,$$

where Q is the subgroup generated by

- $(P, \alpha\circ\beta) - (P,\alpha) - (P,\beta)$, $\forall P \in \mathrm{Obj}\,\mathscr{C}$, $\forall \alpha,\beta \in \mathrm{Aut}(P)$,
- $(M,\beta) - (L,\alpha) - (N,\gamma)$, where the following diagram commutes with exact lines :

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "\alpha"'] & M \arrow[r, "p"] \arrow[d, "\beta"] & N \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & L \arrow[r, "i"] & M \arrow[r, "p"] & N \arrow[r] & 0 \end{tikzcd}.$$

3.3.2 Remark. The relations in $K_1\mathscr{C}$ imply that

- $[(P,\mathrm{id}_P)] = [(P,\mathrm{id}_P\circ\mathrm{id}_P)] = 2[(P,\mathrm{id}_P)]$, and so $[(P,\mathrm{id}_P)] = 0$,
- $[(P,\alpha)] = -[(P,\alpha^{-1})]$.

3.3.3 Proposition. $K_1R \cong K_1\mathscr{F}(R)$.

Proof. • We construct a homomorphism $K_1R \longrightarrow K_1\mathscr{F}(R)$. We use the universal property of K_1R and the generalized determinant $\{s_n : \mathrm{GL}_n(R) \longrightarrow K_1R\}_{n\in\mathbb{N}^*}$. We need to find a generalized determinant $\{\delta_n : \mathrm{GL}_n(R) \longrightarrow K_1\mathscr{F}(R)\}_{n\in\mathbb{N}^*}$. For al $A \in \mathrm{GL}_n(R)$ there is an associated homomorphism :

$$\begin{aligned} \lambda_A : R^{\oplus n} &\longrightarrow R^{\oplus n} \\ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} &\longmapsto A\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}. \end{aligned}$$

So it makes sense to define

$$\begin{aligned} \delta_n : \mathrm{GL}_n(R) &\longrightarrow K_1\mathscr{F}(R) \\ A &\longmapsto [(R^{\oplus n}, \lambda_A)]. \end{aligned}$$

This is at least a well defined function. We now check the axioms :

1. $\delta_n(AB) = \delta_n(A)\delta_n(B)$, easily.

2. $\delta_n(\tau_{i,j}(r)) = 0$. Consider

$$\partial_j : R^{\oplus(n-1)} \longrightarrow R^{\oplus n}$$
$$\begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} \longmapsto \begin{pmatrix} v_1 \\ \vdots \\ v_{i-1} \\ 0 \\ v_i \\ \vdots \\ v_{n-1}. \end{pmatrix}$$

Remark that the following diagram commutes with exact lines :

$$\begin{tikzcd} 0 \arrow[r] & R^{\oplus(n-1)} \arrow[r, "\partial_j"] \arrow[d, "\mathrm{id}_{R^{\oplus(n-1)}}"'] & R^{\oplus n} \arrow[r, "\mathrm{proj}_j"] \arrow[d, "\lambda_{\tau_{i,j}(r)}"] & R \arrow[r] \arrow[d, "\mathrm{id}_R"] & 0 \\ 0 \arrow[r] & R^{\oplus(n-1)} \arrow[r, "\partial_j"] & R^{\oplus n} \arrow[r, "\mathrm{proj}_j"] & R \arrow[r] & 0 \end{tikzcd}.$$

Thus

$$\begin{aligned} [(R^{\oplus n}, \lambda_{\tau_{i,j}(r)})] &= [(R^{\oplus(n-1)}, \mathrm{id}_{R^{\oplus(n-1)}})] + [(R, \mathrm{id}_R)] \\ &= 0. \end{aligned}$$

3. $\delta_{n+1}\left(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\right) = \delta_n(A)$. Note $B = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Consider

$$\begin{tikzcd} 0 \arrow[r] & R^{\oplus n} \arrow[r, "\partial_{n+1}"] \arrow[d, "\lambda_A"'] & R^{\oplus(n+1)} \arrow[r, "\mathrm{proj}_{n+1}"] \arrow[d, "\lambda_B"] & R \arrow[r] \arrow[d, "\mathrm{id}_R"] & 0 \\ 0 \arrow[r] & R^{\oplus n} \arrow[r, "\partial_{n+1}"] & R^{\oplus(n+1)} \arrow[r, "\mathrm{proj}_{n+1}"] & R \arrow[r] & 0 \end{tikzcd}.$$

Thus

$$\begin{aligned} [(R^{\oplus(n+1)}, \lambda_B)] &= [(R^{\oplus n}, \lambda_A)] + [(R, \mathrm{id}_R)] \\ &= [(R^{\oplus n}, \lambda_A)]. \end{aligned}$$

So by the niversal property of K_1R, there exists a unique homomorphism $\widehat{\delta} : K_1R \longrightarrow K_1\mathscr{F}(R)$ such that $\widehat{\delta} \circ s_n = \delta_n$, i.e.

$$\widehat{\delta}\left(\begin{pmatrix} A & 0 \\ 0 & I_\infty \end{pmatrix}\right) = \delta_n(A).$$

- We now define the homomorphism $K_1\mathscr{F}(R) \longrightarrow K_1R$. Define a function

$$\mathrm{Iso}\{(P, \alpha) \mid P \in \mathrm{Obj}\,\mathscr{F}(R), \alpha \in \mathrm{Aut}(P)\} \longrightarrow K_1R$$

as follows : choose a basis for P, i.e. choose an isomorphism of R-modules $\varepsilon_P : P \xrightarrow{\cong} R^{\oplus n}$. Define

$$\varepsilon((P,\alpha)) = s_n(\varepsilon_P \circ \alpha \circ \varepsilon_P^{-1})$$

seen as a n by n matrix. Let $\widetilde{\varepsilon} : F_{\mathbf{Ab}}\,\mathrm{Iso}\{\cdots\} \longrightarrow K_1R$ be the unique homomorphism defined by ε. We need to show that $Q \subseteq \ker\widetilde{\varepsilon}$.

- Consider $P \in \mathrm{Obj}\,\mathscr{F}(R)$ and automorphisms $\alpha, \beta \in \mathrm{Aut}(P)$. Observe that

$$\begin{aligned}
\widetilde{\varepsilon}((P,\alpha\circ\beta)) &= s_n(\varepsilon_P \circ \alpha \circ \beta \circ \varepsilon_P^{-1})\\
&= s_n(\varepsilon_P \circ \alpha \circ \varepsilon_P^{-1} \circ \varepsilon_P\beta \circ \varepsilon_P^{-1})\\
&= s_n(\varepsilon_P \circ \alpha \circ \varepsilon_P^{-1}) s_n(\varepsilon_P\beta \circ \varepsilon_P^{-1})\\
&= \widetilde{\varepsilon}((P,\alpha))\widetilde{\varepsilon}((P,\beta))
\end{aligned}$$

  Therefore $[(P,\alpha\circ\beta)] - [(P,\alpha)] - [(P,\beta)] \in \ker\varepsilon$.

- If the following diagram commutes with exact lines

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "\alpha"'] & M \arrow[r, "p"] \arrow[d, "\beta"] & N \arrow[r] \arrow[d, "\gamma"] & 0\\
0 \arrow[r] & L \arrow[r, "i"] & M \arrow[r, "p"] & N \arrow[r] & 0
\end{tikzcd},$$

  then there is a choice of isomorphisms (or equivalently a choice of bases) such that

$$\begin{tikzcd}
L \arrow[r, "\varepsilon_L", "\cong"'] \arrow[d] & R^{\oplus l} \arrow[d]\\
M \arrow[r, "\varepsilon_L \oplus \varepsilon_N", "\cong"'] \arrow[d] & R^{\oplus m} = R^{\oplus l} \oplus R^{\oplus n} \arrow[d]\\
N \arrow[r, "\varepsilon_N", "\cong"'] & R^{\oplus n}
\end{tikzcd}.$$

  Then

$$\begin{aligned}
&\widetilde{\varepsilon}((M,\beta) - (L,\alpha) - (N,\gamma))\\
&= \widetilde{\varepsilon}((M,\beta))\widetilde{\varepsilon}((L,\alpha))^{-1}\widetilde{\varepsilon}((N,\gamma))^{-1}\\
&= s_{l+n}(\varepsilon_M \circ \beta \circ \varepsilon_M^{-1}) s_l(\varepsilon_L \circ \alpha^{-1} \circ \varepsilon_L^{-1}) s_n(\varepsilon_N \circ \gamma^{-1} \circ \varepsilon_N^{-1})\\
&= s_{l+n}\left(\begin{pmatrix} \varepsilon_L \circ \alpha \circ \varepsilon_L^{-1} & 0\\ 0 & \varepsilon_N \circ \gamma \circ \varepsilon_N^{-1}\end{pmatrix}\begin{pmatrix} \varepsilon_L \circ \alpha^{-1} \circ \varepsilon_L^{-1} & 0\\ 0 & I_n\end{pmatrix}\begin{pmatrix} I_l & 0\\ 0 & \varepsilon_N \circ \gamma^{-1} \circ \varepsilon_N^{-1}\end{pmatrix}\right)\\
&= s_{l+n}(I_{l+n})\\
&= I_\infty.
\end{aligned}$$

Finally, $Q \subseteq \ker\widetilde{\varepsilon}$. Therefore, there exists a unique homomorphism

$$\begin{aligned}
\widehat{\varepsilon} : K_1\mathscr{F}(R) &\longrightarrow K_1R\\
[(P,\alpha)] &\longmapsto s_n(\varepsilon_P \circ \alpha \circ \varepsilon_P^{-1}
\end{aligned}$$

- It is not hard to show that $\widehat{\delta}$ and $\widehat{\varepsilon}$ are mutually inverse. So $K_1\mathscr{F}(R) \cong K_1R$.

□

3.3.4 Proposition. $K_1\mathscr{F}(R) = K_1\mathscr{P}(R)$.

Proof. Exercise set 13. □

3.4 K-Theory as a homology theory of rings

The idea here is to explore the analogies with homology theories of topological spaces.

3.4.1 Definition (Excision)**.** Let R and R' be two rings, and $J \subseteq R$ and $J' \subseteq R'$ be two sided ideals. An **excision** with respect to J and J' is a homomorphism $\phi : R \longrightarrow R'$ that restricts and corestricts to an isomorphism $\phi|_J^{J'} : J \xrightarrow{\cong} J'$.

3.4.2 Definition (Relative K-Theory groups)**.** Consider the following pullback

$$\begin{tikzcd} D(R,J) = R \times_{R/J} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & R \arrow[d, "q"] \\ R \arrow[r, two heads, "q"] & R/J \end{tikzcd},$$

where $D(R,J) = \{(r_1, r_2) \in R^2 \mid q(r_1) = q(r_2)\}$ is the **double** of R with respect to J. Then we define the **relative K-Theory groups** :

$$K_0(R,J) = K_0D(R,J)$$
$$K_1(R,J) = K_1D(R,J).$$

3.4.3 Theorem (Excision)**.** An excision $\phi : R \longrightarrow R'$ with respact to J and J' induces a isomorphisms

$$K_0(R,J) \cong K_0(R',J')$$
$$K_1(R,J) \cong K_1(R',J').$$

"Away from J and J', the rings look the same".

3.4.4 Theorem (Mayer–Vietoris)**.** For every pair of ring homomorphisms $R \xrightarrow{\phi} T \xleftarrow{\pi} S$, where π is surjective, there is an exact sequence

$$\begin{tikzcd} K_1(R \times_T S) \arrow[r, "K_1 \operatorname{proj}_R \oplus K_1 \operatorname{proj}_S"] & K_1R \oplus K_1S \arrow[r, "K_1\phi - K_1\pi"] & K_1T \arrow[dll, "\partial"] \\ K_0(R \times_T S) \arrow[r, "K_0 \operatorname{proj}_R \oplus K_0 \operatorname{proj}_S"] & K_0R \oplus K_0S \arrow[r, "K_0\phi - K_0\pi"] & K_0T \end{tikzcd}$$

where $R \otimes_T S = \{(r,s) \in R \times S \mid \phi(r) = \pi(s)\}$.

3.4.5 Remark. The localization sequence extends to K_1 : for all commutative ring R, $\forall S \subseteq R$ that does not contains 0 nor zero-divisors, there exists an exact sequence :

$$\begin{tikzcd} & K_1R \arrow[r] & K_1S^{-1}R \arrow[dll, out=0, in=180] \\ K_0\mathscr{T}^2_{R,S} \arrow[r] & K_0R \arrow[r] & K_0S^{-1}R \end{tikzcd}.$$

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