Representation of the Symmetric Groups

H. Geranios
TEXed by Marion Jeannin and Cédric Ho Thanh

Fall 2013

Contents

1 Tableaux and partitions **1**
1.1 Definitions 1
1.2 Dominance 3

2 Tabloids and permutations modules **5**
2.1 Definitions 5
2.2 Character of M^λ 7

3 Polytabloids and Specht modules **9**
3.1 Definition 9
3.2 Irreducibility of S^λ 10
3.3 Basis of S^λ 14
3.4 Basis of $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}(S^\lambda, M^\mu)$ 19

4 Symmetric functions **25**
4.1 Definitions 25
4.2 Elementary symmetric functions 27
4.3 Complete symmetric functions 27
4.4 Schur functions 29

5 Induced modules **33**

6 Character of the Specht modules **37**
6.1 The hook formula 41

Introduction

Definition. This document is unofficial lecture notes from the course *Representation of the Symmetric Groups*, given by H. Geranios during the fall semester 2013.

Corollary. *This document is provided as is, without warranty of any kind. Don't hesitate to spot mistakes so I can correct them.*

Chapter 1

Tableaux and partitions

1.1 Definitions

We denote by $\mathfrak{S}_n$ the symmetric group on n elements.

Reminders 1.1.1. 1. If $(i_1 \cdots i_k), \pi \in \mathfrak{S}_n$, then

$$\pi^{-1}(i_1 \cdots i_k)\pi = (\pi(i_1) \cdots \pi(i_k)).$$

2. Each element of $\mathfrak{S}_n$ can be decomposed into a product of disjoint cycles in a unique way (up the order of the terms). Moreover, disjoint cycles commutes.

If $\sigma, \rho \in \mathfrak{S}_n$ are conjugated and have disjoint cycles decompositions $\sigma = \sigma_1 \cdots \sigma_k$ and $\rho = \rho_1 \cdots \rho_l$, sorted by decreasing length, then clearly each σ_i is conjugated to a ρ_j, and conversely. There is a one-to-one correspondence between the disjoints cycles in the decomposition of σ and those of ρ. Moreover, conjugates of σ_i have the same length of σ_i. Therefore we come to the following result :

Proposition 1.1.2. *Let $\sigma, \rho \in \mathfrak{S}_n$ be two permutations having disjoint cycles decompositions $\sigma = \sigma_1 \cdots \sigma_k$ and $\rho = \rho_1 \cdots \rho_l$. If σ and ρ are conjugated, then $k = l$ and $\{length\ \sigma_i\}_{i \leq k} = \{length\ \rho_i\}_{i \leq l}$, seen as multisets.*

We introduce now a more convenient object to work with.

Definition 1.1.3 (Partition)**.** Let $n \in \mathbb{N}$. A *partition* of n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ satisfying

1. $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$,
2. $\sum_{i=1}^{k} \lambda_i = n$.

We adopt the convention that $(\lambda_1, \ldots, \lambda_k) = (\lambda_1, \ldots, \lambda_k, 0, 0, \ldots)$, that is, we can add 0s without changing the partition. We note $\lambda \vdash n$ and by $P(n)$ the set of all partitions of n.

Take $\sigma \in \mathfrak{S}_n$. As disjoint cycles commutes, we can define $\lambda(\sigma)$ to be the list of lengths of the disjoints cycles in the decomposition of σ plus the trivial permutations (i), sorted in decreasing order. We have that $\lambda(\sigma)$ is a partition of n, which we call the *type* of σ.

Corollary 1.1.4. *Two permutations are conjugate iff they have the same type.*

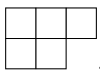
Corollary 1.1.5. *The number of conjugacy classes of $\mathfrak{S}_n$ is $|P(n)|$.*

Remark 1.1.6. Let $\sigma \in \mathfrak{S}_n$. Consider $\lambda(\sigma) = (\lambda_1, \ldots, \lambda_k)$, and denote $m_i = |\{j \mid \lambda_j = i\}|$. By combinatorial observations :

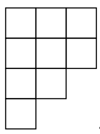
$$|\operatorname{Cl}(\sigma)| = n! \prod_{i=1}^{k} \frac{1}{m_i! i_i^m}.$$

Definition 1.1.7 (Young diagram)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$. Then the *Young diagram* $[\lambda]$ of λ is the shape obtained by the following method : put λ_i boxes in the ith row.

Examples 1.1.8. 1. Consider $\lambda = (3, 2) \vdash 5$. Then

$[\lambda] =$  .

2. Consider $\lambda = (3, 3, 2, 1) \vdash 9$. Then

$[\lambda] =$  .

Definition 1.1.9 (Young subgroup)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$. The *Young subgroup* of $\mathfrak{S}_n$ corresponding λ is

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,\ldots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots .$$

Remark that $\mathfrak{S}_\lambda \cong \prod_{\lambda_i \in \lambda} \mathfrak{S}_{\lambda_i}$.

Example 1.1.10. If $\lambda = (3, 3, 2, 1) \vdash 9$, then

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,2,3\}} \times \mathfrak{S}_{\{4,5,6\}} \times \mathfrak{S}_{\{7,8\}} \times \mathfrak{S}_{\{9\}}.$$

Definition 1.1.11 (Young tableau)**.** Let $\lambda \vdash n$. A *(Young-) tableau* is a bijection between the Young diagram $[\lambda]$ and the set $\{1, \ldots, n\}$. In other terms, it consists in a filling of the Young diagram $[\lambda]$ with the elements of $\{1, \ldots, n\}$.

Example 1.1.12. Consider $\lambda = (3,3,2,1) \vdash 9$. Then

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & \\ \cline{1-2} 9 & & \\ \cline{1-1} \end{array} \quad \text{and} \quad S = \begin{array}{|c|c|c|} \hline 3 & 7 & 2 \\ \hline 6 & 9 & 1 \\ \hline 4 & 5 & \\ \cline{1-2} 8 & & \\ \cline{1-1} \end{array}$$

are both Young tableaux of shape λ.

1.2 Dominance

Definition 1.2.1 (Dominance of partitions)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_k), \mu = (\mu_1, \ldots, \mu_k) \vdash n$. We say that λ *dominates* μ if

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i, \qquad \forall j \leq k.$$

We note $\mu \trianglelefteq \lambda$.

Examples 1.2.2. 1. We have $(3,3) \trianglelefteq (4,2)$.

2. Consider $\lambda = (3,3) = (3,3,0)$ and $\mu = (4,1,1)$. Then $\lambda \ntrianglelefteq \mu$ and $\mu \ntrianglelefteq \lambda$. Hence, $\trianglelefteq$ doesn't form a total order.

Definition 1.2.3 (Total dominance)**.** We denote by $\leq$ the lexicographical order. In other words, let $\lambda = (\lambda_1, \ldots, \lambda_k), \mu = (\mu_1, \ldots, \mu_k) \vdash n$. We note $\mu \leq \lambda$ if either

1. $\lambda = \mu$,

2. $\mu_j < \lambda_j$, where j is the first index where λ and μ differ.

Lemma 1.2.4. *Let $\lambda, \mu \vdash n$. Then*

$$\mu \trianglelefteq \lambda \implies \mu \leq \lambda.$$

Proof. Assume that $\lambda \neq \mu$, and let j be the first index where λ and μ differ. Then :

$$\begin{aligned} \mu \trianglelefteq \lambda &\implies \sum_{i=1}^{j} \mu_i \leq \sum_{i=1}^{j} \lambda_i \\ &\implies \mu_j \leq \lambda_j. \end{aligned}$$

□

Lemma 1.2.5 (Dominance lemma for partition)**.** *Let $\lambda, \mu \vdash n$ and T, S be tableaux of shape λ and μ. If the numbers (elements) in each row of S appear in different columns in T, then $\mu \trianglelefteq \lambda$.*

Proof. Lift'em up ! For instance, consider, $\lambda = (4,2)$, $\mu = (3,3)$, and tableaux

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 5 & 6 & 3 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \qquad \text{and} \qquad S = \begin{array}{|c|c|c|}\hline 4 & 6 & 2 \\ \hline 1 & 5 & 3 \\ \hline\end{array}.$$

The first line of S is $(4,6,2)$. The 6 is already in the first line of T. As $(4,6,2)$ lies in different comumns (by hypothesis), we can "lift" the 2 and the 4 up to the first line of T :

$$T' = \begin{array}{|c|c|c|c|}\hline 2 & 4 & 6 & 3 \\ \hline 1 & 5 \\ \cline{1-2}\end{array}, \qquad S = \begin{array}{|c|c|c|}\hline 4 & 6 & 2 \\ \hline 1 & 5 & 3 \\ \hline\end{array}.$$

Now, each number of the first line of S is in the first line of T. Hence, if we sum the numbers, we'll get a greater or equal result in T'.

$$T' = \begin{array}{|c|c|c|c|l}\cline{1-4} 2 & 4 & 6 & 3 & \quad \Sigma = 15 \\ \cline{1-4} 1 & 5 \\ \cline{1-2}\end{array}, \qquad S = \begin{array}{|c|c|c|l}\cline{1-3} 4 & 6 & 2 & \quad \Sigma = 12 \\ \cline{1-3} 1 & 5 & 3 \\ \cline{1-3}\end{array}.$$

This shows that the first line of T' is longer (or of the same length) than the first line in S, and hence $\lambda_1 \geq \mu_1$. We continue by lifting up the numbers in the second line of S to the first two lines of T', which again is possible by hypothesis, we sum up, and get $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$. We would continue on with larger tableaux, but the result will always be that $\mu \trianglelefteq \lambda$. □

Chapter 2

Tabloids and permutations modules

2.1 Definitions

Let $\lambda \vdash n$. Denote by $\mathrm{Tab}(\lambda)$ the set of all tableaux of shape λ. We then have an obvious action

$$\mathfrak{S}_n \times \mathrm{Tab}(\lambda) \longrightarrow \mathrm{Tab}(\lambda)$$

which permutes the numbers in the tableaux. For example

$$(1\,3\,6) \cdot \begin{array}{|c|c|c|}\hline 4 & 6 & 2 \\ \hline 1 & 5 & 3 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 & 6 \\ \hline \end{array}.$$

Let T be a tableau of shape $\lambda \vdash n$. We define

- $R(T)$ to be the subgroup of $\mathfrak{S}_n$ that consists of permutations that stabilize the rows of T,
- $C(T)$ to be the subgroup of $\mathfrak{S}_n$ that consists of permutations that stabilize the columns of T.

Lemma 2.1.1. *Let T be a tableau of shape $\lambda \vdash n$, and let $\pi \in \mathfrak{S}_n$. Then*

1. $R(\pi T) = \pi R(T) \pi^{-1}$,
2. $C(\pi T) = \pi C(T) \pi^{-1}$.

Proof. Let $\pi \in \mathfrak{S}_n$. We can write it as a product of transposition. So we just have to check the lemma for transposition, and it is then very easy. □

We define an equivalence relation on $\mathrm{Tab}(\lambda)$ by

$$T \sim T' \iff \exists \pi \in R(T) \text{ such that } \pi T = T'.$$

Equivalence classes are calles *tabloid* of shape λ. The equivalence class of T is denoted by $\{T\}$. For instance

$$T = \begin{array}{|c|c|c|c|}\hline 1&5&6&3\\ \hline 2&4 \\ \cline{1-2}\end{array} \implies \{T\} = \begin{array}{|cccc|}\hline 1&5&6&3\\ \hline 2&4 \\ \cline{1-2}\end{array} = \begin{array}{|cccc|}\hline 1&3&5&6\\ \hline 2&4 \\ \cline{1-2}\end{array} = \cdots .$$

We have an action of $\mathfrak{S}_n$ on the set of tabloid $\mathrm{Tab}(\lambda)/\sim$:

$$\begin{aligned} \mathfrak{S}_n \times \mathrm{Tab}(\lambda)/\sim &\longrightarrow \mathrm{Tab}(\lambda)/\sim \\ (\pi, \{T\}) &\longmapsto \{\pi T\}. \end{aligned}$$

It is well defined. Indeed, let $T, T' \in \mathrm{Tab}(\lambda)$ be such that $\{T\} = \{T'\}$. Then by definition, there exists $\sigma \in R(T)$ such that $\sigma T = T'$. We have that $\rho = \pi\sigma\pi^{-1} \in R(\pi T)$, as $R(\pi T) = \pi R(t)\pi^{-1}$. Then

$$\begin{aligned} \rho\pi T &= \pi\sigma\pi^{-1}\pi T \\ &= \pi\sigma\sigma T \\ &= \pi T', \end{aligned}$$

and so $\{\pi T\} = \{\pi T'\}$.

Definition 2.1.2 (Permutation module)**.** Let $\lambda \vdash n$. Define the *permutation module* M^λ corresponding to the partition λ by

$$M^\lambda = \mathbb{C}[\mathrm{Tab}(\lambda)/\sim].$$

Definition 2.1.3 (Cyclic G-module)**.** Let G be a group, and V be a G-module. It is said cyclic as a G-module if it is a cyclic module over the ring $\mathbb{C}G$. In other words, $V = \mathbb{C}\{gv \mid g \in G\}$ for a $v \in V$. Remark that then

$$x = \sigma_i \lambda_i g_i v, \qquad \forall x \in V.$$

Proposition 2.1.4. *The permutation module M^λ is cyclic, and*

$$M^\lambda \cong \mathbb{C}[\mathfrak{S}_n/\mathfrak{S}_\lambda].$$

Proof. Remark that $\mathrm{Tab}(\lambda)/\sim$ is a transitive $\mathfrak{S}_n$-set. It is then clear that M^λ is cyclic. Take any $\{T\} \in \mathrm{Tab}(\lambda)/\sim$. Then $\mathrm{Stab}_{\mathfrak{S}_n}(\{T\}) \cong \mathfrak{S}_\lambda$ (recall the definition of the Young subgroup), and $\mathrm{Orb}(\{T\}) = \mathrm{Tab}(\lambda)/\sim$, since, again, the action is transitive. We have $\mathrm{Orb}(\{T\}) \cong \mathfrak{S}_n/\mathfrak{S}_\lambda$ as sets, and so

$$\begin{aligned} M^\lambda &= \mathbb{C}[\mathrm{Tab}(\lambda)/\sim] && \text{by definition} \\ &\cong \mathbb{C}\,\mathrm{Orb}(\{T\}) \\ &\cong \mathbb{C}[\mathfrak{S}_n/\mathfrak{S}_\lambda]. \end{aligned}$$

□

Corollary 2.1.5. *If $\lambda = (\lambda_1, \ldots, \lambda_k)$, then*

$$\dim M^\lambda = \frac{n!}{\prod_i \lambda_i!}.$$

2.2 Character of M^λ

We now try to compute the character of M^λ. Let $\pi \in \mathfrak{S}_n$. It is clear that

$$\chi_{M^\lambda}(\pi) = \sharp\{\{T\} \mid \{\pi T\} = \{T\}\}.$$

That's not convenient... Consiter λ to be of length n (just add 0's). Take $\pi \in \mathfrak{S}_n$. The aim of what's folloging is still to compute the number of fixed tabloids by the action of π. Let $\{T\}$ be such a tabloid, and $(x_1 \cdots x_q)$ be a q-cycle of π. Then all the numbers x_i lies in the same row of T. Hence, we can "arrange" those cycles among rows of T. If π has m_q cycles of length q, and if $r_{p,q}$ is the number of q-cycle at row p then we have

$$\frac{m_q!}{\prod_{i=1}^n r_{i,q}!}$$

ways to permute those cycles among the rows of the tabloid. Therefore,

$$\chi_{M^\lambda}(\pi) = \sum_X \frac{m_q!}{\prod_{i=1}^n r_{i,q}!},$$

where $X = \{(r_{p,q})_{p,q} \mid \sum_{i=1}^n r_{i,q} = m_q \text{ and } \sum_{i=1}^n r_{p,i} = \lambda_p\}$ describes all the way of arranging the cycles of π among the rows of T. Remark that

$$\prod_{q=1}^n (X_1^q + \cdots + X_n^q)^{m_q} = \sum_Y \frac{m_q!}{\prod_{i=1}^n r_{i,q}!} X_1^{r_{1,q}} X_2^{2r_{2,q}} \cdots X_n^{nr_{n,q}},$$

where $Y = \{(r_{p,q})_{p,q} \mid \sum_{i=1}^n r_{i,q} = m_q\}$. So $\chi_{M^\lambda}(\pi)$ is the coefficient of $X_1^{\lambda_1} \cdots X_n^{\lambda_n}$ in the above polynomial !

Example 2.2.1. Take $\lambda = (2,2), \mu = (2,1,1) \vdash 4$. The partition μ determines the following polynomial :

$$(X_1^2 + X_2^2 + X_3^2 + X_4^2)(X_1 + X_2 + X_3 + X_4)^2,$$

in which the monomial $X_1^2 X_2^2$ has coefficient 2. So $\chi_{M^\lambda}(\mu) = 2$ (for this notation, recall that each partition of n completely determine a conjugacy class of $\mathfrak{S}_n$).

Chapter 3

Polytabloids and Specht modules

3.1 Definition

Take a subset $H \subseteq \mathfrak{S}_n$, and define

$$\begin{aligned} a_H &= \sum_{\sigma \in H} \sigma \in \mathbb{C}\mathfrak{S}_n, \\ b_H &= \sum_{\sigma \in H} \operatorname{sgn}(\sigma)\sigma = \sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma)\sigma \in \mathbb{C}\mathfrak{S}_n. \end{aligned}$$

Let T be a tableau, and consider $C(T)$. Define

$$\begin{aligned} k_T &= b_{C(T)} \in \mathbb{C}\mathfrak{S}_n, \\ e_T &= k_T\{T\} \in M^\lambda. \end{aligned}$$

Example 3.1.1. Take

$$T = \begin{array}{|c|c|c|} \hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2} \end{array} .$$

We have that

$$\begin{aligned} C(T) &= \mathfrak{S}_{\{3,4\}} \times \mathfrak{S}_{\{1,5\}} \times \mathfrak{S}_{\{2\}}, \\ k_T &= 1 - (3\,4) - (1\,5) + (3\,4)(1\,5), \\ e_T &= \begin{array}{|ccc|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array} - \begin{array}{|ccc|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} - \begin{array}{|ccc|} \hline 2 & 4 & 5 \\ \hline 1 & 3 \\ \cline{1-2} \end{array} + \begin{array}{|ccc|} \hline 2 & 3 & 5 \\ \hline 1 & 4 \\ \cline{1-2} \end{array} . \end{aligned}$$

Lemma 3.1.2. *If $\pi \in \mathfrak{S}_n$, take any tableau T of shape $\lambda \vdash n$. Then*

1. $k_{\pi T} = \pi k_T \pi^{-1}$,

2. $e_{\pi T} = \pi e_T$.

Proof. 1. We have :

$$\begin{aligned} k_{\pi T} &= \sum_{\sigma \in C(\pi T)} \operatorname{sgn}(\sigma)\sigma \\ &= \sum_{\sigma \in \pi C(T)\pi^{-1}} \operatorname{sgn}(\sigma)\sigma \\ &= \sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma)\pi\sigma\pi^{-1} \\ &= \pi k_T \pi^{-1}. \end{aligned}$$

2. We have :

$$\begin{aligned} e_{\pi T} &= k_{\pi T}\{\pi T\} \\ &= \pi k_T \pi^{-1}\{\pi T\} \\ &= \pi e_T. \end{aligned}$$

□

Definition 3.1.3 (Polytabloid, Specht module)**.** We call e_T a *polytabloid*. We note $S^\lambda = \langle e_T \mid T \in \mathrm{Tab}(\lambda)\rangle \leq M^\lambda$ the *Specht module* corresponding to λ.

Lemma 3.1.4. *The submodule S^λ is cyclic.*

Proof. Trivial, since $e_{\pi T} = \pi e_T$. □

Example 3.1.5. Take $\lambda = (1, \ldots, 1) = (1^n) \vdash n$. Then all tabloids have form

$$\begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \hline x_n \\ \hline \end{array}\,.$$

There is $n!$ of them, so $\dim M^\lambda = n!$.

3.2 Irreducibility of S^λ

Define an inner product on M^λ by

$$\langle \{T\}, \{T'\} \rangle = \begin{cases} 1 & \text{if } \{T\} = \{T'\} \\ 0 & \text{otherwise,} \end{cases} \qquad \forall T, T' \in \mathrm{Tab}(\lambda),$$

which we extend by linearity.

Lemma 3.2.1. *The inner product $\langle -, - \rangle$ is $\mathfrak{S}_n$-invariant.*

Proof. Trivial. □

Lemma 3.2.2 (The sign lemma)**.** *Let $H \leq \mathfrak{S}_n$,*

1. *if $\pi \in H$, then $\pi b_H = b_H \pi = \operatorname{sgn}(\pi) b_H$,*
2. *if $u, v \in M^\lambda$, then $\langle b_H u, v\rangle = \langle u, b_H v\rangle$,*
3. *if $(i\ j) \in H$, then $b_H = \kappa(\mathrm{id} - (i\ j))$, for some $\kappa \in \mathbb{C}\mathfrak{S}_n$,*
4. *if $(i j) \in H$ and if i and j are in the same row of a tableau $T \in \mathrm{Tab}(\lambda)$, then $b_H\{T\} = 0$.*

Proof. 1. We have

$$\begin{aligned}
\pi b_H &= \pi \sum_{\sigma \in H} \operatorname{sgn}(\sigma)\sigma \\
&= \sum_{\sigma \in H} \operatorname{sgn}(\sigma)\pi\sigma \\
&= \sum_{\sigma \in H} \operatorname{sgn}(\pi)\operatorname{sgn}(\pi\sigma)\pi\sigma \qquad \text{since } \operatorname{sgn}(\pi)^2 = 1 \\
&= \operatorname{sgn}(\pi) b_H.
\end{aligned}$$

We do similarly to show that $b_H \pi = \operatorname{sgn}(\pi) b_H$.

2. If $u, v \in M^\lambda$, then

$$\begin{aligned}
\langle b_H u, v\rangle &= \sum_{\sigma \in H} \operatorname{sgn}(\sigma)\langle \sigma u, v\rangle \\
&= \sum_{\sigma \in H} \operatorname{sgn}(\sigma)\langle u, \sigma^{-1} v\rangle \\
&= \langle u, b_H v\rangle.
\end{aligned}$$

3. If $(i\ j) \in H$, then $K = \{\mathrm{id}, (i\ j)\} \leq H$, and there exists a set of representative $\pi_1, \ldots, \pi_r \in H$ such that

$$H = \coprod_{k=1}^{r} \pi_k K.$$

Hence :

$$\begin{aligned}
b_H &= \sum_{\sigma \in H} \operatorname{sgn}(\sigma)\sigma \\
&= \underbrace{\sum_{k=1}^{r} \operatorname{sgn}(\pi_k)\pi_k}_{\kappa}(\mathrm{id} - (i\ j)).
\end{aligned}$$

4. We have

$$\begin{aligned} b_H\{T\} &= \kappa(\mathrm{id} - (i\, j))\{T\} \\ &= \kappa(\{T\} - \{(i\, j)T\}) \\ &= 0. \end{aligned}$$

□

Lemma 3.2.3. *Take $\lambda, \mu \vdash n$, $T \in \mathrm{Tab}(\lambda)$ and $T' \in \mathrm{Tab}(\mu)$, then*

1. *if $k_T\{T'\} \neq 0$, then $\mu \trianglelefteq \lambda$,*
2. *if $k_T\{T'\} \neq 0$ and $\lambda = \mu$, then $k_T\{T'\} = \pm e_T$.*

Proof. 1. Pick i and j in the same row of T'. Then they cannot be in the same column of T, and hence, by dominance lemma, $\mu \trianglelefteq \lambda$.

2. We have that T and T' have the same shape. There exists $\pi \in \mathfrak{S}_n$ such that $T' = \pi T$. Since $k_T\{T'\} \neq 0$, we have that $\pi \in C(T)$. Hence :

$$\begin{aligned} k_T\{T'\} &= k_T\{\pi T\} \\ &= \sum_{\sigma \in C(T)} \mathrm{sgn}(\sigma)\sigma\{\pi T\} \\ &= \mathrm{sgn}(\pi) e_T. \end{aligned}$$

□

Corollary 3.2.4. *If $u \in M^\lambda$, and $T \in \mathrm{Tab}(\lambda)$, then $k_T u = c e_T$, for a $c \in \mathbb{C}$.*

Proof. Since $u \in M^\lambda$, we can write $u = \sum_i c_i\{T_i\}$, and so

$$\begin{aligned} k_T u &= \sum_i c_i \underbrace{k_T}_{=0 \text{ or } \pm e_T} \{T_i\} \\ &= c e_T. \end{aligned}$$

□

Theorem 3.2.5 (Submodule theorem)**.** *If $U \leq M^\lambda$ is a submodule, then either $S^\lambda \leq U$ or $U \leq (S^\lambda)^\perp$.*

Proof. If $u \in U$ and $T \in \mathrm{Tab}(\lambda)$, then $k_T u = c e_T \in U$. Therefore, if $\exists u \in U$ and $\exists T \in \mathrm{Tab}(\lambda)$ such that $k_T u \neq 0$, then $c \neq 0$. So we have that $c e_T \in U$, and so $e_T \in U$. As S^λ is generated by e_T, $\forall T \in \mathrm{Tab}(\lambda)$, we have $S^\lambda \leq U$.

Assume now that $\forall u \in U$, $\forall T \in \text{Tab}(\lambda)$, $k_T u = 0$. Then we have that $\langle u, e_T \rangle = 0$, $\forall u \in U$, $\forall T \in \text{Tab}(\lambda)$. Indeed,

$$\begin{aligned}\langle u, e_T \rangle &= \langle u, k_T \{T\} \rangle \\ &= \langle k_T u, \{T\} \rangle \\ &= 0.\end{aligned}$$

So $U \leq (S^\lambda)^\perp$. □

Corollary 3.2.6. *The Specht-modules S^λ are irreducible.*

But are they isomorphic ?

Lemma 3.2.7. *Let $\theta : S^\lambda \longrightarrow M^\mu$, for $\lambda, \mu \vdash n$.*

1. *If $\theta \neq 0$, then $\mu \trianglelefteq \lambda$.*
2. *If $\lambda = \mu$, then $\theta = c\,\text{id}_{S^\lambda}$, for a $c \in \mathbb{C}$.*

Proof.

1. We know that $\exists e_T \in M^\lambda$ such that $\theta(e_T) \neq 0$. Since $M^\lambda = S^\lambda \oplus (S^\lambda)^\perp$, we can extend θ into $\hat{\theta}$ by agreeing that $\hat{\theta}|_{(S^\lambda)^\perp} = 0$. Hence

$$\begin{aligned}\theta(e_T) &= \theta(k_T \{T\}) \\ &= k_T \hat{\theta}(\{T\}) \\ &\in M^\mu.\end{aligned}$$

So $0 \neq \hat{\theta}(\{T\}) = \sum_i c_i \{T_i\}$, where $T_i \in \text{Tab}(\mu)$. So there exist at least one T_i such that $k_T\{T_i\} \neq 0$. So $\mu \trianglelefteq \lambda$.

2. As above, we extend θ into $\hat{\theta} : M^\lambda \longrightarrow M^\lambda$. We have $\theta(e_T) = \sum_i c_i \{T_i\}$, where $T_i \in \text{Tab}(\lambda)$. Then, $\theta(k_T\{T\}) = k_T \hat{\theta}(\{T\}) = c e_T$. Then

$$\begin{aligned}\theta(e_{\pi T}) &= \theta(\pi e_T) \\ &= \theta(\pi k_T \{T\}) \\ &= \pi k_T \hat{\theta}(\{T\}) \\ &= \pi c e_T \\ &= c e_{\pi T}.\end{aligned}$$

So $\theta(u) = cu$, $\forall u \in S^\lambda$. Since the result holds for all e_T (which are the generators of S^λ).

□

Lemma 3.2.8. *The set $\{S^\lambda \mid \lambda \vdash n\}$ is a full set of irreducible modules of $\mathfrak{S}_n$.*

Proof. We just have to prove that they are pairwise non isomorphic. Assume $\phi : S^\lambda \xrightarrow{\cong} S^\mu$ is a non zero isomorphism. We extend it into $\hat{\phi} : M^\lambda \longrightarrow S^\mu \longrightarrow M^\mu$. So $\hat{\phi} \neq 0$, and $\mu \trianglelefteq \lambda$. On the other hand, $\widehat{\phi^{-1}} : M^\mu \longrightarrow M^\lambda$ is non zero either, and so $\lambda \trianglelefteq \mu$. Finally, $\lambda = \mu$, and $S^\lambda = S^\mu$. □

Corollary 3.2.9. *Let $\lambda \vdash n$. Consider the following decomposition into irreducible modules :*

$$M^\lambda = \bigoplus_{\mu \vdash n} (S^\mu)^{\oplus m_{\lambda,\mu}}.$$

Then $m_{\lambda,\mu} = 0$ if $\mu \not\trianglerighteq \lambda$. Moreover, $m_{\lambda,\lambda} = 1$.

Proof. Recall that

$$\begin{aligned} m_{\lambda,\mu} &= \langle \chi_{S^\mu}, \chi_{M^\lambda} \rangle \\ &= \dim \operatorname{Hom}(S^\mu, M^\lambda). \end{aligned}$$

If $m_{\lambda,\mu} > 0$, then there exists a non zero morphism $\theta : S^\mu \longrightarrow M^\lambda$, which indices a non zero morphism $\hat{\theta} : M^\mu \longrightarrow M^\lambda$, and so $\lambda \trianglelefteq \mu$. Moreover,

$$\begin{aligned} m_{\lambda,\lambda} &= \langle S^\lambda, M^\lambda \rangle \\ &= \dim \operatorname{Hom}(S^\lambda, M^\lambda) \\ &= 1, \end{aligned}$$

by a previous result. □

3.3 Basis of S^λ

We call a tableau $T \in \mathrm{Tab}(\lambda)$ *standard* if the entries of T are sorted increasingly in all its rows and all its columns.

Examples 3.3.1. Consider $\lambda = (4, 2) \vdash 6$, then

1	5	6	3
2	4		

is not standard, whereas

1	2	4	6
3	5		

is.

Definitions 3.3.2 (From exercise set 5)**.** 1. A *composition* of an integer n is a sequence of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_k)$, such that $\sum_{i=1}^k \lambda_i = n$. The integers λ_i are the *parts of the composition*.

2. Let λ and μ be two composition of n. We say that λ *dominates* μ, which is denoted by $\mu \trianglelefteq \lambda$, if

$$\sum_{i=1}^{j} \mu_i \leq \sum_{i=1}^{j} \lambda_i, \qquad \forall j \leq k.$$

3. Let $\{T\}$ be a tabloid of shape $\lambda \vdash n$. For each $i \leq n$, we define the composition λ^i if i by

$$\lambda^i_j = \text{ the number of entries that are } \leq i \text{ in the } j\text{-th row of } \{T\}.$$

The sequence of compositions $(\lambda^1, \dots, \lambda^n)$ is called the *composition sequence* of $\{T\}$.

4. Let $\{T\}$ and $\{T'\}$ be two tabloids of shape $\lambda \vdash n$, with corresponding composition sequences $(\lambda^1, \dots, \lambda^n)$ and $(\mu^1, \dots, \mu^n)$ respectively. We say that $\{T\}$ *dominates* $\{T'\}$, which is denoted by $\{T'\} \trianglelefteq \{T\}$, if

$$\mu^i \trianglelefteq \lambda^i, \qquad \forall i \leq n.$$

Reminder 3.3.3. Let $(A, \leq)$ be a poset.

1. An element $a \in A$ is a *maximum* if $b \leq a$, $\forall b \in A$.
2. An element $a \in A$ is *maximal* if $a \leq b \implies a = b$, $\forall b \in A$.

Lemma 3.3.4 (Dominance lemma for tabloids)**.** *Let$\lambda \vdash n$, and $T \in \mathrm{Tab}(\lambda)$. If $k < l \leq n$, and k appears in a lower row (that is, drawn lower) than l in $[T]$, then*

$$\{T\} \triangleleft (k\,l)\{T\}.$$

Proof. Exercise 4 from sheet 5. □

Corollary 3.3.5. *If T is a standard tableau, and if $\{S\}$ appears in e_T, then $\{S\} \triangleleft \{T\}$.*

Lemma 3.3.6. *Let $v_1, \dots, v_m \in M^\lambda$. Suppose that each v_i admit $\{T^i\}$ as maximum tabloid in their decomposition, and that those $\{T^i\}$s are distincts. Then $v_1, \dots, v_m$ are linearly independent.*

Proof. Without loss of generality, we can assume that $\{T^1\}$ is maximal among all tabloids. Then $\{T^1\}$ cannot appear in v_i, $\forall i \neq 1$, or it would otherwise be maximum in v_i, and so $\{T^1\} = \{T^i\}$. Suppose that $v_1 = d\{T^1\} + \sum_j d_j\{T_j\}$. Let $c_1, \dots, c_m \in \mathbb{C}$. Then

$$\begin{aligned}\sum_{i=1}^{m} c_i v_i = 0 &\implies \sum_{i=1}^{m} c_i\{T^i\} = 0 \\ &\implies c_1 d\{T^1\} + \sum_j c_1 d_j\{T_j\} + \sum_{i=2}^{m} c_i v_i = 0 \\ &\implies c_1 = 0,\end{aligned}$$

as $\{T_1\}$ only appears once, with coefficient $c_1 d$, and $d \neq 0$. So $\sum_{i=2}^{m} d_i c_i \{T^i\} = 0$, and we conclude by induction. $\square$

Proposition 3.3.7. *The set* $\{e_T \mid T$ *standard of shape* $\lambda\} \subseteq S^\lambda$ *is linearly independent.*

Proof. Remark that a standard tabloid $\{T\}$ is maximum among the terms of e_T. We can conclude using the previous lemma. $\square$

We now show that the standard polytabloids generate S^λ, that is, generate any other e_T, where T isn't standard. We can always assume that the columns of T are increasing from top to bottom, as this property is always true up to column stabilizer : if $T \in \mathrm{Tab}(\lambda)$ doesn't have the required property, then $\exists \sigma \in C(T)$ such that σT does. Moreover,

$$\begin{aligned} e_{\sigma T} &= \sigma e_T \\ &= \sigma \sum_{\tau \in C(T)} \mathrm{sgn}(\tau)\{\tau T\} \\ &= \mathrm{sgn}(\sigma) e_T. \end{aligned}$$

So if T isn't standard, we can assume that the order problems occur in the rows of T.

Definition 3.3.8 (Garnir element)**.** Let A and B be two disjoint finite sets. Consider the permutations groups $\mathfrak{S}_A$, $\mathfrak{S}_B$, and $\mathfrak{S}_{A\cup B}$. We have that $\mathfrak{S}_A \times \mathfrak{S}_B \leq \mathfrak{S}_{A\cup B}$. Let $\pi_1, \ldots, \pi_r$ be a family of representatives of the left cosets of $\mathfrak{S}_A \times \mathfrak{S}_B$:

$$\mathfrak{S}_{A\cup B} = \biguplus_{i=1}^{r} \pi_i(\mathfrak{S}_A \times \mathfrak{S}_B),$$

where $\uplus$ stands for the disjoint union, and where $\pi_1 = \mathrm{id}$. We define a *garnir element* of (A, B) to be

$$g_{A,B} = \sum_{i=1}^{r} \mathrm{sgn}(\pi_i)\pi_i \in \mathbb{C}\mathfrak{S}_{A\cup B}.$$

This element depends on the choice of representatives, and is therefore not unique !

The group $\mathfrak{S}_{A\uplus B}$ acts obviously on the set of all possible ordered pairs (A', B') satisfying $|A'| = |A|$, $|B'| = |B|$, and $A' \uplus B' = A \uplus B$. For every such pair (A', B'), assume that $\pi_{(A',B')}$ is such that

$$\pi_{(A',B')}(A, B) = (A', B').$$

The $\pi_{(A',B')}$ form a family of representatives of the cosets of $\mathfrak{S}_A \times \mathfrak{S}_B$.

Definition 3.3.9 (Garnir element of a tableau)**.** Let $\lambda \vdash n$, $T \in \mathrm{Tab}(\lambda)$, $j < n$, A a set of elements of the jth column of T, and B a set of elements in the $j+1$th column of T. Then a *garnir element* of T is a garnir element of the pair (A, B), choosing the representatives π such that the elements of $A \uplus B$ are sorted in increasing order in πT.

Example 3.3.10. Consider $(3, 2, 1) \vdash 6$, and

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 & 3 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} .$$

The we can define $A = \{5, 6\}$ and $B = \{2, 3\}$:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 & 3 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} .$$

The representatives satisfying the above condition are $\{\mathrm{id}, (2\,3\,5), (2\,3\,6\,5)\}$. Indeed :

$$(2\,3\,5)T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} ,$$

$$(2\,3\,6\,5)T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} .$$

The corresponding garnir element is then

$$g_{A,B} = \mathrm{id} + (2\,3\,5) - (2\,3\,6\,5).$$

Proposition 3.3.11. *If T is a tableau, choose A and B as before. If $|A \uplus B|$ is greater than the number of elements in the jth column, then $g_{A,B} e_T = 0$.*

Proof. We first show that $b_{\mathfrak{S}_{A \uplus B}} e_T = 0$. Let $\sigma \in C(T)$. Then there are $a, b \in A \uplus B$ such that a and b appear in the same row of σT. Then, $(a\, b) \in \mathfrak{S}_{A \uplus B}$, and so by previous lemma, $b_{\mathfrak{S}_{A \uplus B}} \sigma\{T\} = 0$, which leads to our assertion. We know that

$$\mathfrak{S}_{A \uplus B} = \biguplus_{i=1}^{r} \pi_i (\mathfrak{S}_A \times \mathfrak{S}_B),$$

and so $b_{\mathfrak{S}_{A \uplus B}} = g_{A,B} b_{\mathfrak{S}_A \times \mathfrak{S}_B}$. Using our assertion, we have that $g_{A,B} b_{\mathfrak{S}_A \times \mathfrak{S}_B} e_T = 0$. However, $\mathfrak{S}_A \times \mathfrak{S}_B \leq C(T)$, and so

$$\begin{aligned} 0 &= g_{A,B} b_{\mathfrak{S}_A \times \mathfrak{S}_B} e_T \\ &= g_{A,B} |\mathfrak{S}_A \times \mathfrak{S}_B| e_T. \end{aligned}$$

□

Definition 3.3.12. Let $T \in \mathrm{Tab}(\lambda)$. We define the *column tabloid* to be the orbit of T under the action of $C(T)$. De denote this set by $[T]$. We define dominance for such tabloids in a similar fashion that for row tabloids.

Example 3.3.13. Let $(2,1) \vdash 3$, and

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} \; .$$

Then

$$[T] = \begin{array}{|c|c|}\hline 1 & 3 \\ 2 & \\ \cline{1-1}\end{array} \; .$$

Proposition 3.3.14. *The set* $\{e_T \mid T \in \mathrm{Tab}(\lambda),\ T$ *standard of shape* $\lambda\}$ *generates* S^λ.

Proof. We only show that $\{e_T \mid T \in \mathrm{Tab}(\lambda),\ T \text{ standard of shape } \lambda\}$ generates $\{e_T \mid T \in \mathrm{Tab}(\lambda),\ T \text{ of shape } \lambda\}$. If T is a tableau such that $[T]$ is maximal, then T shares its tabloid $[T]$ with a standard tableaux, and so e_T is generated by the standard polytabloids. If T doesn't have such a property, assume by induction hypothesis that any tableau S such that $[T] \trianglelefteq [S]$ does. We can assume without loss of generality that T has sorted comumns (from top to bottom). As $[T]$ isn't standard, the violations of standardness occur in the rows. Pick the first one (that is the highest row in the drawing that present such a violation). Here is a little drawing of T :

$\cdots$	a_1	b_1	$\cdots$
$\cdots$	$\vdots$	$\vdots$	$\cdots$
$\cdots$	$a_i \geq$	b_i	$\cdots$
$\cdots$	$\vdots$	$\vdots$	$\cdots$
$\cdots$	$\vdots$	b_q	$\cdots$
$\cdots$	$\vdots$		
$\cdots$	a_p		

.

Define $A = \{a_i, \ldots, a_p\}$, and $B = \{b_i, \ldots, b_q\}$. Since $|A \uplus B| \geq p$, we know by a previous proposition that $g_{A,B} e_T = 0$. Then :

$$g_{A,B} e_T = e_T + \sum_{i=2}^{r} \mathrm{sgn}(\pi_i)\pi_i e_T = 0 \qquad \text{remember : } \pi_1 = \mathrm{id}$$

$$\Longrightarrow \; e_T = -\sum_{i=2}^{r} \mathrm{sgn}(\pi_i)\pi_i e_T = -\sum_{i=2}^{r} \mathrm{sgn}(\pi_i) e_{\pi_i T}.$$

Since $[T] \trianglelefteq [\pi_i T]$, $\forall i \geq 2$, the result follows by decreasing induction. $\square$

Corollary 3.3.15. *1. The set $\{e_T \mid T \in \mathrm{Tab}(\lambda), T$ standard of shape $\lambda\}$ is a basis of S^λ.*

2. If $f^{(\lambda)} = \dim S^\lambda$ is the number of standard tableaux of shape λ, then

$$n! = \sum_{\lambda \vdash n} (f^{(\lambda)})^2.$$

3.4 Basis of $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}(S^\lambda, M^\mu)$

Definitions 3.4.1 (From exercise sheet 6)**.** 1. A *generalized Young tableau* of shape $\lambda \vdash n$ is a Young diagram of shape λ filled from numbers from 1 to n, allowing repetitions.

2. The *type* or *content* of a generalized Young tableau T is the composition μ, where μ_i is the number of i entry in T.

3. Define
$$\mathcal{T}_{\lambda,\mu} = \{T \mid T \text{ has shape } \lambda \text{ and content } \mu\}.$$

4. A generalized tableau is said *semistandard* if its rows are weakly increasing, and its columns strictly increasing (from top to bottom).

5. Define
$$\mathcal{T}^0_{\lambda,\mu} = \{T \in \mathcal{T}_{\lambda,\mu} \mid T \text{ is semistandard}\}.$$

6. Define the *Kostka number* to be
$$\mathcal{K}_{\lambda,\mu} = |\mathcal{T}^0_{\lambda,\mu}|.$$

Example 3.4.2. This

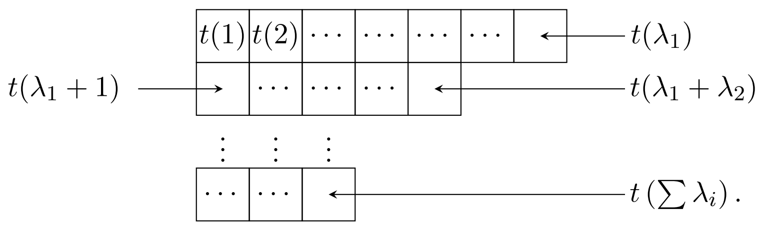
$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

is a nonsemistandard generalized tableau of shape $(2,1)$ and content $(0,2,1)$. This

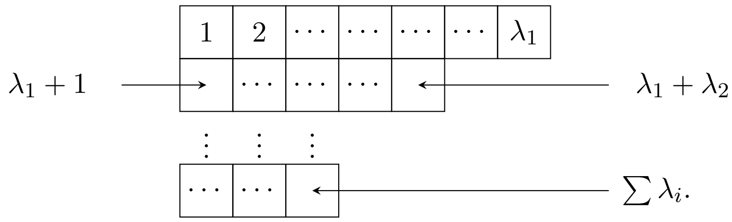
$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

is a nonsemistandard tableau of shape $(2,1)$ and content $(2,0,1)$.

Let $t \in \mathcal{T}_{\lambda,\mu}$. Denote by $t(i)$ the i-th entry of t :



From now on, denote T_0 be the tabeau such that $T_0(i) = i$, i.e. it is of the following form :



Let S be a tabloïd of shape μ. We define $t \in \mathcal{T}_{\lambda,\mu}$ from $\{S\}$ by :

$$t(i) = \text{ the index of the row in which } i \text{ appears in } \{S\}.$$

Example 3.4.3. Choose $\lambda = (3, 2)$ and $\mu = (2, 2, 1)$. We have

$$\{S\} = \begin{array}{|cc|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline 5 & \\ \hline \end{array} \implies t = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 & 3 & \\ \hline \end{array}.$$

This construction defines a bijection from tabloids of shape μ to $\mathcal{T}_{\lambda,\mu}$, which we extend into an isomorphism of vector space

$$\Theta : M^\mu \longrightarrow \mathbb{C}\mathcal{T}_{\lambda,\mu}.$$

We want this to be a $\mathbb{C}\mathfrak{S}_n$-isomorphism, and so we check that it is compatible with the action of $\mathfrak{S}_n$. We have an action

$$\begin{aligned} \mathfrak{S}_n \times T_{\lambda,\mu} &\longrightarrow \mathcal{T}_{\lambda,\mu} \\ (\pi, t) &\longmapsto \pi t, \end{aligned}$$

where $\pi t(i) = t(\pi^{-1}(i))$. Let $\{S\} \in M^\mu$. We have

$$\begin{aligned} \Theta(\pi\{S\})(i) &= \text{ the index of the row in which } i \text{ appears in } \pi\{S\} \\ &= \text{ the index of the row in which } \pi^{-1}(i) \text{ appears in } \{S\} \\ &= \Theta(\{S\})(\pi^{-1}(i)) \\ &= \pi\Theta(\{S\}). \end{aligned}$$

So Θ is a $\mathbb{C}\mathfrak{S}_n$-isomorphism.

We now take interest in $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}(M^\lambda, M^\mu)$. Let $t \in \mathcal{T}_{\lambda,\mu}$. We can define the tabloids $\{t\}$ and $[t]$ in a similar fashion as we did with tableaux. Define a homomorphism

$$\begin{aligned} \Theta_t : M^\lambda &\longrightarrow \mathbb{C}\mathcal{T}_{\lambda,\mu} \cong M^\mu \\ \{T_0\} &\longmapsto \sum_{S \in \{t\}} S. \end{aligned}$$

This defines Θ_t completely since M^λ can be generated only by T_0 as $[\sigma_n]$-module. Remark that this construction makes Θ_t into a $\mathbb{C}\mathfrak{S}_n$-homomorphism. Denote $\bar{\Theta}_t = \Theta_t|_{S^\lambda}$. We have

$$\begin{aligned}\bar{\Theta}_t(e_{T_0}) &= \bar{\Theta}_t(k_{T_0}\{T_0\})\\ &= k_{T_0}\Theta_t(\{T_0\})\\ &= k_{T_0}\sum_{S\in\{t\}} S.\end{aligned}$$

Proposition 3.4.4. *Let T_0 be our fixed tableau of shape λ, and take $t \in \mathcal{T}_{\lambda,\mu}$. Then $k_{T_0}t = 0$ iff t has two equal elements in the same column.*

Proof. $\implies$ If $k_{T_0}t = 0$, then

$$t + \sum_{\pi\in C(T_0)\setminus\{\mathrm{id}\}} \mathrm{sgn}(\pi)\pi t = 0.$$

As the generalized tableaux form a basis, this zero linear combination forces the term $-t$ to appear in $\sum_{\pi\in C(T_0)\setminus\{\mathrm{id}\}} \mathrm{sgn}(\pi)\pi t$. So $\exists\sigma \in C(T_0) \setminus \{\mathrm{id}\}$ such that $\mathrm{sgn}(\sigma) = -1$, and such that $\sigma t = t$. As σ is a column stabilizer, this forces t to have at least two equal elements in a same column.

$\impliedby$ Suppose $t(i) = t(j)$, with the i-th and j-th entries in the same column. We have $(i\ j) \in C(T_0)$. By the sign lemma, $\exists k \in \mathbb{C}\mathfrak{S}_n$ such that $k_{T_0} = k(\mathrm{id} - (i\ j))$. So

$$\begin{aligned}k_{T_0}t &= kt - k(i\ j)t\\ &= kt - kt\\ &= 0.\end{aligned}$$

□

Recall that $\mathcal{K}_{\lambda,\mu} = 0 \implies \mu \lhd \lambda$. We define the dominance order on the generalized tabloids and on the generalized column tabloids in the same fashion as we did with tabloids.

Lemma 3.4.5 (Dominance lemma for generalized column tabloids)**.** *If $t \in \mathcal{T}_{\lambda,\mu}$, k and l appears in the i-th and j-th row respectively, $i < j$, $k < l$, then $[t] \lhd [(k\ l)t]$.*

Let V be a $\mathbb{C}$-vector space, and $B = \{b_1, \ldots, b_m\}$ one of its basis. Suppose that $\sim$ is an equivalence relation on V, and that $\leq$ is a partial order on $V/\sim$.

Lemma 3.4.6. *Let $v_1, \ldots, v_n \in V$, and assume that*

1. *each v_i has a maximal component in b_i,*

2. *$[b]_\sim \leq [b_i]_\sim$, for all b in which v_i has a nonzero component.*

Then $v_1, \ldots, v_n$ are linearly independent.

Proof. Exercise 7.2. □

Lemma 3.4.7. *Let V and W be two $\mathbb{C}$-vector spaces. If $\Theta_1, \ldots, \Theta_n \in \mathrm{Hom}_{\mathbb{C}}(V, W)$, and if there exists $v \in V$ such that $\Theta_1(v), \ldots, \Theta_n(v)$ are linearly independent in W. Then $\Theta_1, \ldots, \Theta_n$ are linearly independent in $\mathrm{Hom}_{\mathbb{C}}(V, W)$.*

Proof. Trivial. Indeed if $\sum_{i=1}^n \alpha_i \theta_i = 0$ then $\sum_{i=1}^n \alpha_i \theta_i(v) = 0$, therefore $\alpha_i = 0\ \forall i$. □

Proposition 3.4.8. *The set $\{\bar{\Theta}_t \mid t \in \mathcal{T}^0_{\lambda,\mu}\}$ is linearly independent in $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}(S^\lambda, M^\mu)$.*

Proof. Suppose that $\mathcal{T}^0_{\lambda,\mu} = \{t_1, \ldots, t_m\}$. In the light of the previous lemmas, we only prove that $\bar{\Theta}_{t_1}(e_{T_0}), \ldots, \bar{\Theta}_{t_m}(e_{T_0})$ are linearly independent in M^μ. We have

$$\begin{aligned}\bar{\Theta}_{t_i}(e_{T_0}) &= \bar{\Theta}_{t_i}(k_{T_0}\{T_0\}) \\ &= k_{T_0}\Theta_{t_i}(\{T_0\}) \\ &= k_{T_0} \sum_{S \in \{t_i\}} S.\end{aligned}$$

Since t_i is semistandard, we have that $[S] \lhd [t_i]$, $\forall S \in \{t_i\}$, $S \neq t_i$, by dominance lemma. Hence $[t_i]$ is the maximum term of $k_{T_0}\Theta_{t_i}(\{T_0\})$ with a nonzero coefficient. Moreover, the t_is are distinct. So by previous lemma, the $\bar{\Theta}_{t_i}(e_{T_0})$ are linearly independent. □

Lemma 3.4.9. *Assume that $\Theta \in \mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}(S^\lambda, M^\mu)$ is such that*

$$\Theta(e_{T_0}) = \sum c_t t.$$

Then

1. *If $\pi \in C(T_0)$ and $t_1 = \pi t_2$, then $c_{t_1} = \mathrm{sgn}(\pi) c_{t_2}$.*

2. *If t has a repeat in a column, then $c_t = 0$.*

3. *If $\Theta \neq 0$, then there is a semistandard tableau t such that $c_t \neq 0$.*

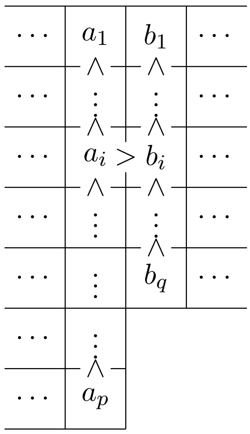
Proof. 1. We have

$$\begin{aligned}\pi\Theta(e_{T_0}) &= \Theta(\pi e_{T_0})\\ &= \Theta(\mathrm{sgn}(\pi)e_{T_0})\\ &= \mathrm{sgn}(\pi)\Theta(e_{T_0}).\end{aligned}$$

The result follows.

2. Assume that t has a repeat un a column, say at entries i and j. We have $(i\,j)t = t$, and so $c_t = -c_t = 0$.

3. Assume that $\Theta \neq 0$. There is a $c_{t_2} \neq 0$, and we can take it to be maximal for this property. By previous lemma, we can take t_2 to have increasing columns, and since t_2 doesn't have any repeats, we can take it to have strictly increasing columns. Any violation is semistandardness occurs in the rows.



Take $A = \{a_i, \dots, a_p\}$, and $B = \{b_1, \dots, b_i\}$. Consider a garnir element $g_{A,B}$ with id beeing one of its term. We have $g_{A,B}e_{T_0} = 0$, and so

$$\begin{aligned}g_{A,B}\sum c_t t &= g_{A,B}\Theta(e_{T_0})\\ &= \Theta(g_{A,B}e_{T_0})\\ &= 0.\end{aligned}$$

We have that $g_{A,B}t_2 = t_2 + \sum \cdots$. There is a generalized tableau S and a permutation π appearing in $g_{A,B}$ such that $\pi S = t_2$, and so $[t_2] \lhd [S]$, a contradiction with the maximality of t_2.

□

Proposition 3.4.10. *The set* $\{\bar{\Theta}_t \mid t \in \mathcal{T}^0_{\lambda,\mu}\}$ *spans* $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}(S^\lambda, M^\mu)$.

Proof. Let $\Theta \in \mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}(S^\lambda, M^\mu)$, such that

$$\Theta(e_{T_0}) = \sum c_t t.$$

Define $L_\Theta = \{S \in \mathcal{T}^0_{\lambda,\mu} \mid [S] \trianglelefteq [t] \text{ for some term } t \text{ of } \Theta(e_{T_0})\}$. By induction on $|L_\Theta|$:

- If $|L_\Theta| = 0$, then $\Theta = 0$.
- If $|L_\Theta| > 0$, then $\Theta \neq 0$, and we can take t_2 to be the semistandard tableau appearing in $\Theta(e_{T_0})$ with $[t_2]$ maximal. Take

  $$\Theta_2 = \Theta - c_{t_2}\bar{\Theta}_{t_2}.$$

  Remark that $L_{\Theta_2} \subseteq L_\Theta$. Moreover, $t_2 \in L_\Theta \setminus L_{\Theta_2}$ (exercise), and so $|L_{\Theta_2}| < |L_\Theta|$. The result follows by induction.

□

Corollary 3.4.11. *The set $\{\bar{\Theta}_t \mid t \in \mathcal{T}^0_{\lambda,\mu}\}$ is a basis of $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}(S^\lambda, M^\mu)$.*

Corollary 3.4.12 (Young's rule)**.** *We have*

$$M^\mu \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus \mathcal{K}_{\lambda,\mu}}.$$

Proof. If $m_{\lambda,\mu}$ is the coefficient of S^λ in M^μ, then

$$\begin{aligned} m_{\lambda,\mu} &= \dim \mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}(S^\lambda, M^\mu) \\ &= |\mathcal{T}^0_{\lambda,\mu}| \\ &= \mathcal{K}_{\lambda,\mu}. \end{aligned}$$

□

Chapter 4

Symmetric functions

4.1 Definitions

Let's start with $\mathbb{Z}[X_1, \ldots, X_n]$. We have an obvious action

$$\begin{aligned} \mathfrak{S}_n \times \mathbb{Z}[X_1, \ldots, X_n] &\longrightarrow \mathbb{Z}[X_1, \ldots, X_n] \\ (\sigma, f(X_1, \ldots, X_n)) &\longmapsto f(X_{\sigma(1)}, \ldots, X_{\sigma(n)}). \end{aligned}$$

We call a polynomial *symmetric* if $\sigma f(X_1, \ldots, X_n) = f(X_1, \ldots, X_n)$, $\forall \sigma \in \mathfrak{S}_n$, and denote by Λ_n the subring of $\mathbb{Z}[X_1, \ldots, X_n]$ of symmetric polynomials. We have a grading

$$\mathbb{Z}[X_1, \ldots, X_n] = \bigoplus_{k \geq 0} \mathbb{Z}[X_1, \ldots, X_n]_k,$$

where $\mathbb{Z}[X_1, \ldots, X_n]_k$ is the subgroup of homogeneous polynomials of degree k. Wa obtain an induced grading

$$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k,$$

where Λ_n^k is the subgroup of $\mathbb{Z}[X_1, \ldots, X_n]_k$ of homogeneous symmetric polynomials of degree k. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$. Denote $X^\alpha = X^{\alpha_1} \cdots X^{\alpha_n}$. Take λ a partition (not necessarily of n) that has length $l(\lambda) \leq n$. Define

$$m_\lambda(X_1, \ldots, X_n) = \sum_{\sigma \in \mathfrak{S}_n | Stab(\lambda)} X^{\sigma\lambda},$$

where σ runs through $\mathfrak{S}_n$ such that $\sigma\lambda$ doesn't repeat. It is clear that, $m_\lambda(X_1, \ldots, X_n)$ are symmetric polynomials. Moreover, $\lambda \neq \mu$ implies $m_\lambda(X_1, \ldots, X_n) \neq m_\mu(X_1, \ldots, X_n)$.

Take $f(X_1, \ldots, X_n)$ a nonzero symmetric polynomial. Then X^α appears in $f(X_1, \ldots, X_n)$, for some α. Since $f(X_1, \ldots, X_n)$ is symmetric, $X^{\sigma\alpha}$ also appears, $\forall \sigma \in \mathfrak{S}_n$. Take $\tau \in \mathfrak{S}_n$ such that $\tau\alpha = \lambda$ is a partition. We have that $m_\lambda(X_1, \ldots, X_n)$ appears in $f(X_1, \ldots, X_n)$, and so

$$\langle m_\lambda(X_1, \ldots, X_n) \mid l(\lambda) \leq n,\ \lambda \text{ partition}\rangle = \Lambda_n.$$

The later set is moreover a $\mathbb{Z}$-basis. We then have that $\{m_\lambda(X_1, \ldots, X_n) \mid l(\lambda) \leq n,\ \lambda \vdash k\}$ is a basis of Λ_n^k. If $k \leq n$, we can drop the condition that $l(\lambda) \leq n$.

Take $n \leq m$, and consider

$$\begin{aligned} \rho : \mathbb{Z}[X_1, \ldots, X_m] &\longrightarrow \mathbb{Z}[X_1, \ldots, X_n] \\ X_i &\longmapsto \begin{cases} X_i & \text{if } i \leq n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

From that, we build

$$\begin{aligned} \rho_{m,n} : \Lambda_m &\longrightarrow \Lambda_n \\ m_\lambda(X_1, \ldots, X_m) &\longmapsto m_\lambda(X_1, \ldots, X_n), \end{aligned}$$

where $l(\lambda) \leq n$, and

$$\begin{aligned} \rho_{m,n}^k : \Lambda_m^k &\longrightarrow \Lambda_n^k \\ m_\lambda(X_1, \ldots, X_m) &\longmapsto m_\lambda(X_1, \ldots, X_n), \end{aligned}$$

where again, $l(\lambda) \leq n$. If $k \leq n \leq m$, then the later map is a bijection.

Let Λ^k be the set of sequences of the form $(f_i)_{i \in \mathbb{N}}$, where

1. $f_n \in \Lambda_n^k$,
2. $f_m(X_1, \ldots, X_n, 0, \ldots) = f_n(X_1, \ldots, X_n)$, $\forall m \geq n$.

Obviously, Λ^k is a $\mathbb{Z}$-module. Assume that $k \leq n$, and define

$$\begin{aligned} \rho^n : \Lambda^k &\longrightarrow \Lambda_n^k \\ (f_i)_{i \in \mathbb{N}} &\longmapsto f_n. \end{aligned}$$

Remark that this a $\mathbb{Z}$-isomorphism (small exercise). So for every $m_\lambda(X_1, \ldots, X_n)$, there exists $m_\lambda \in \Lambda^k$ such that

$$\rho^n(m_\lambda) = m_\lambda(X_1, \ldots, X_n).$$

Here, m_λ and $m_\lambda(X_1, \ldots, X_n)$ are *two distinct things* that live on different worlds !!! Therefore, Λ^k is free with basis $\{m_\lambda \mid \lambda \vdash n\}$.

Example 4.1.1. Take $\lambda = (4, 1) \vdash 5$. Then

- $m_0 = 0$,
- $m_1 = 0$,
- $m_2 = X_1^4 X_2 + X_1 X_2^4$,
- $m_3 = X_1^4 X_2 + X_1 X_2^4 + X_1^4 X_3 + X_1 X_3^4 + X_2^4 X_3 + X_2 X_3^4$,
- and so on...

4.2 Elementary symmetric functions

Define

$$e_0 = 1,$$
$$e_r = \sum_{i_1 < \cdots < i_r} X_{i_1} \cdots X_{i_r}.$$

Remark that $e_r = m_{(1^r)}$, and therefore is symmetric., and so $e_n \in \Lambda$. If λ is a partition, say $\lambda = (\lambda_1, \ldots, \lambda_p)$, then define $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_p}$.

Proposition 4.2.1. *1. The set $\{e_\lambda \mid \lambda \vdash k\}$ is a $\mathbb{Z}$-basis of Λ^k.*

2. The elements e_r are algebraically independent, and

$$\Lambda = \mathbb{Z}[e_1, \ldots],$$
$$\Lambda_n = \mathbb{Z}[e_1, \ldots, e_n].$$

Proof. 1. Let $\tilde{\lambda}$ be the transposed partition of λ (i.e. the partition corresponding to the transposed Young diagram $[\lambda]^t$). With respect to the lexicographical order, the following monomial is the highest that appears in $e_{\tilde{\lambda}}$:

$$(X_1 \cdots X_{\tilde{\lambda}_1}) \cdots (X_1 \cdots X_{\tilde{\lambda}_q} = X_1^{\lambda_1} \cdots X_p^{\lambda_p}.$$

Therefore, $m_\lambda(X_1, \ldots, X_n)$ appears in $e_{\tilde{\lambda}}$ with coefficient 1. So $\{e_\lambda \mid \lambda \vdash k\}$ generates $\{m_\lambda(X_1, \ldots, X_n) \mid \lambda \vdash k\}$ and so Λ^k. Moreover, they are algebraically independent, again by the fact that each $m_\lambda(X_1, \ldots, X_n)$ appears with coefficient one in and only in $e_{\tilde{\lambda}}$.

2. Immediate from the first point.

□

4.3 Complete symmetric functions

Definition 4.3.1. Let L be a field, S un subset de L and K a subfield of L, S is algebraically free over K (or equivalently its elements are algebraically independant over K) if, for all finite sequence of distinct elements of S $(s_1, \ldots, s_n)$ and all non zero polynomial $P(X_1, \ldots, X_n) \in K[X]$, $P(s_1, \ldots, s_n) \neq 0$.

Define

$$h_0 = 1,$$
$$h_r = \sum_{i_1 \leq \cdots \leq i_r} X_{i_1} \cdots X_{i_r} = \sum_{\lambda \vdash r} m_\lambda.$$

Those are obviously symmetric functions. If λ is a partition, say $\lambda = (\lambda_1, \ldots, \lambda_p)$, then define $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_p}$.

Proposition 4.3.2. *1. The set $\{h_\lambda \mid \lambda \vdash k\}$ is a $\mathbb{Z}$-basis of Λ^k.*

2.

3. The elements h_r are linearly independent, and

$$\begin{aligned}\Lambda &= \mathbb{Z}[h_1, \ldots],\\ \Lambda_n &= \mathbb{Z}[h_1, \ldots, h_n].\end{aligned}$$

Proof. 1. We already know that $\Lambda = \mathbb{Z}[e_1, \ldots]$. Define

$$\begin{aligned}\omega : \Lambda &\longrightarrow \Lambda\\ e_i &\longmapsto h_i,\end{aligned}$$

which we extend by linearity. We prove that ω is a bijection by showing that $\omega^2 = \mathrm{id}_\Lambda$. Consider the elements of $\Lambda[[t]]$:

$$\begin{aligned}E(t) &= \sum_{r\in\mathbb{N}} e_r t^r\\ &= \prod_{r\geq 1}(1 + X_i t),\end{aligned}$$

the later equality is a small combinatorial exercise, and

$$\begin{aligned}H(t) &= \sum_{r\in\mathbb{N}} h_r t^r\\ &= \prod_{i\geq 1}\sum_{r\in\mathbb{N}}(X_i t)^r\\ &= \prod_{i\geq 1}\frac{1}{1 - X_i t}.\end{aligned}$$

Remark that $E(t)H(-t) = 1$, and so t^n has coefficient 0 in $E(t)H(-t) = 1$. So

$$\sum_{r=0}^{n}(-1)^r e_{n-r} h_r = 0.$$

We prove that $\omega(h_r) = e_r$ by induction on r. If $r = 0$, it is clear. If $r = n + 1$, we know that

$$\begin{aligned}\omega\left(\sum_{r=0}^{n}(-1)^r e_{n-r} h_r\right) &= \sum_{r=0}^{n}(-1)^r h_{n-r}\omega(h_r)\\ &= 0.\end{aligned}$$

So

$$0 = \sum_{r=0}^{n+1} (-1)^r h_{n+1-r}\omega(h_r)$$
$$= \sum_{r=0}^{n} (-1)^r h_{n+1-r}\omega(h_r) + (-1)^{n+1} h_0 \omega(h_{n+1}),$$

and it follows from above that $\omega(h_{n+1}) = e_{n+1}$.

2. Immediate from the first point.

□

4.4 Schur functions

Definition 4.4.1 (From exercise sheet 8)**.**

1. Define the rth *power sum* by

$$p_r = \sum_i X_i^r \in \Lambda^r.$$

Define $p_\lambda = \prod_i p_{\lambda_i}$.

2. Define

$$z_\lambda = \prod_{i \geq 1} m_i! i^{m_i}, \in \mathbb{N}$$

where $m_i = \sharp\{\lambda_j \mid \lambda_j = i\}$.

Consider $\mathbb{Z}[X_1, \dots, X_n]$, and take $a = (a_1, \dots, a_n)$. Define

$$\alpha_a = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) X^{\sigma a} \in A_n^{\Sigma_a},$$

where $\sigma a = (a_{\sigma(1)}, \dots, a_{\sigma(n)})$. It is an antisymetric polynomial. Observe that if $a_i = a_j$ for $i \neq j$, then $\alpha_a = 0$. Hence, we can restrict to the cases where $a_1 > \cdots > a_n \geq 0$. Take $a = \lambda + \delta$, where λ is a partition of length at most n. We have $\delta = (n-1, n-2, \dots, 1, 0)$. Hence clearly

$$\alpha_a = \det(X_i^{\lambda_j + n - j})_{i,j}$$
$$= \det(X_i^{a_j})_{i,j}.$$

The polynomial $\alpha_{\lambda+\delta}$ is divided by $X_i - X_j$ in $\mathbb{Z}[X_1, \dots, X_n]$. Hence

$$\left(\prod_{i<j} X_i - X_j \right) | \alpha_{\lambda+\delta}.$$

Moreover,

$$\prod_{i<j} X_i - X_j = \det(X_i^{n-j})_{i,j} \qquad \text{Vandermonde det.}$$
$$= \alpha_\delta.$$

So $\alpha_\delta | \alpha_{\lambda+\delta}$. Define the *Schur polynomial*

$$S_\lambda(X_1, \ldots, X_n) = \frac{\alpha_{\lambda+\delta}}{\alpha_\delta} = \frac{\det(X_i^{\lambda_j+n-j})_{i,j}}{\det(X_i^{n-j})_{i,j}} \in \Lambda_n.$$

This is a symmetric polynomial, as quotient of two antisymmetric polynomials. If $l(\lambda) \leq n$, and $S_\lambda(X_1, \ldots, X_n, 0) = S_\lambda(X_1, \ldots, X_n)$, then define the *Schur function* as

$$S_\lambda = (0, S_\lambda(X_1), S_\lambda(X_1, X_2), \ldots).$$

Proposition 4.4.2. *The set* $\{S_\lambda(X_1, \ldots, X_n) \mid l(\lambda) \leq n\}$ *is a* $\mathbb{Z}$*-basis of* Λ_n.

Proof. Take A_n to be the submodule of the antisymmetric polynomials. If f is antisymmetric, we can write $f = \alpha_\delta f'$, where f' is symmetric. Define

$$\begin{aligned} \Phi : \Lambda_n &\longrightarrow \Lambda_n \\ f &\longmapsto \alpha_\delta f. \end{aligned}$$

This is a $\mathbb{Z}$-isomorphism. We have that $\{\alpha_{\lambda+\delta} \mid l(\lambda) \leq n\}$ is a $\mathbb{Z}$-basis of A_n. Moreover,

$$\begin{aligned} \Phi(S_\lambda) &= \alpha_\delta S_\lambda \\ &= \alpha_{\lambda+\delta}. \end{aligned}$$

So $\{S_\lambda(X_1, \ldots, X_n) \mid l(\lambda) \leq n\}$ is also a $\mathbb{Z}$-basis of Λ_n. $\square$

Corollary 4.4.3. 1. *The Schur functions are a* $\mathbb{Z}$*-basis of* Λ.

2. *The set* $\{S_\lambda \mid l(\lambda) = k\}$ *is a* $\mathbb{Z}$*-basis of* Λ^k.

Recall that $\mathbb{Z}[e_1, \ldots] = \mathbb{Z}[h_0, \ldots]$. Take $S_\lambda \in \mathbb{Z}[e_1, \ldots] = \mathbb{Z}[h_0, \ldots]$.

Proposition 4.4.4. *Assume that* $l(\lambda) \leq n$*, and* $l(\tilde{\lambda}) \neq m$.

1. *(Jacobi–Trudi, 1841) We have*

$$S_\lambda = \det(h_{\lambda_i - i + j})_{i,j}.$$

2. *(Giambelli, 1903) We have*

$$S_\lambda = \det(e_{\tilde{\lambda}_i - i + j})_{i,j}.$$

Proof. 1. For $1 \le k \le n$, denote by e_r^k the elementary r-symmetric polynomial on $\{X_1, \dots, \widehat{X_k}, \dots, X_n\}$. Let $a = (a_1, \dots, a_n)$,

$$\begin{aligned} M &= \left((-1)^{n-i} e_{n-i}^k\right)_{i,k}, \\ A_a &= (X_j^{a_i})_{i,j}, \\ H_a &= (h_{a_i-n+j})_{i,j}, \\ E^k(t) &= \sum_{r=0}^{n-1} e_r^k t^r = \pi_{i \ne k} 1 + X_i t. \end{aligned}$$

We know that

$$H(t)E(t) = \frac{1}{1 - X_k t}.$$

Consider the coefficient of t^{a_i} :

$$\sum_{j=1}^{n} (-1)^{n_j} h_{a_i-n+j} e_{n-j}^k = X_k^{a_i}.$$

So $H_a M = A_a$, which shows that

$$\underbrace{\det(H_a)}_{=1} \det(M) = \underbrace{\det(A_a)}_{=\alpha_a}.$$

Therefore, $\det(M) = \alpha_a$. Take now $\alpha = \lambda + \delta$. We obtain

$$\det(H_{\lambda+\delta})\alpha_\delta = \alpha_{\lambda+\delta},$$

and so $\det(H_{\lambda+\delta}) = S_\lambda$.

2. Exercise.

□

Example 4.4.5. Take $\lambda = (4, 1)$. Then

$$S_{(4,1)} = \begin{vmatrix} h_4 & h_5 \\ h_0 & h_1 \end{vmatrix} = h_4 h_1 - h_5.$$

Lemma 4.4.6. *We have*

$$\det\left(\frac{1}{1 - X_i Y_j}\right)_{i,j=1}^{n} = \frac{\left(\prod_{i<j} X_i - X_j\right)\left(\prod_{i<j} Y_i - Y_j\right)}{\prod_{i,j} 1 - X_i Y_j}.$$

Lemma 4.4.7. *We have*

$$\prod_{i,j} \frac{1}{1 - X_i Y_j} = \sum_\lambda S_\lambda(X) S_\lambda(Y).$$

Consider the previous automorphism $\omega : \Lambda \longrightarrow \Lambda$. We have

$$\begin{aligned}\omega(S_\lambda) &= \omega(\det(h_{\lambda_i - i + j})_{i,j}) \\ &= \det(e_{\lambda_i - i + j})_{i,j} \\ &= S_{\tilde{\lambda}}.\end{aligned}$$

Define

$$\begin{aligned}\langle -, - \rangle : \Lambda \times \Lambda &\longrightarrow \mathbb{Z} \\ (h_\lambda, m_\mu) &\longmapsto \delta_{\lambda,\mu},\end{aligned}$$

which we extend by linearity.

Theorem 4.4.8.

1. $\langle S_\lambda, S_\mu \rangle = \delta_{\lambda,\mu}$.
2. $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}$.

Chapter 5

Induced modules

Assume that G is a group, and that $H \leq G$ is a subgroup. If V is a $\mathbb{K}G$-module, then $V|_H$ is the restricted $\mathbb{K}H$. Conversely, if W is a $\mathbb{K}H$-module, can we construct a $\mathbb{K}G$-module out of it ? Let $\{s_1, \ldots, s_r\}$ be the representatives of the left cosets G/H. Write

$$G = \coprod_{i=1}^{r} s_i H.$$

Each element of G is of the form $g = s_i h$, for some $1 \leq i \leq r$, and $h \in H$. Consider $V = W^{\oplus r}$. We have that G acts on V by :

$$\begin{aligned} G \times V &\longrightarrow V \\ (g, (w_1, \ldots, w_r)) &\longmapsto (h_{g,\sigma_g^{-1}(1)} w_{\sigma_g^{-1}(1)}, \ldots, h_{g,\sigma_g^{-1}(r)} w_{\sigma_g^{-1}(r)}). \end{aligned}$$

Denote by $W^G = \mathrm{Ind}_H^G(W) = V$. We can embed W into V in the following way :

$$\begin{aligned} W &\hookrightarrow V \\ w &\longmapsto (w, 0, \ldots, 0). \end{aligned}$$

We have

$$\mathbb{K}G = \bigoplus_{i=1}^{r} \mathbb{K} s_i H,$$

and

$$\begin{aligned} \mathbb{K}G \otimes_{\mathbb{K}H} W &\cong \bigoplus_{i=1}^{r} \mathbb{K} s_i H \otimes_{\mathbb{K}H} W \\ &\cong \bigoplus_{i=1}^{r} \underbrace{s_i \otimes W}_{\{s_i \otimes w | w \in W\}} \quad . \end{aligned}$$

So every element of $\mathbb{K}G \otimes_{\mathbb{K}H} W$ is of the form $x = \sum_{i=1}^r s_i \otimes w_i$. Moreover

$$\begin{aligned} g\sum_{i=1}^r s_i \otimes w_i &= \sum_{i=1}^r (gs_i) \otimes w_i \\ &= \sum_{i=1}^r s_{\sigma_g(i)} h_{g,i} \otimes w_i \\ &= \sum_{i=1}^r s_{\sigma_g(i)} \otimes (h_{g,i} w_i) \\ &= \sum_{i=1}^r s_i \otimes h_{g,\sigma_g^{-1}(i)} w_{\sigma_g^{-1}(i)}. \end{aligned}$$

Hence, we have a $\mathbb{K}G$-isomorphism

$$\begin{aligned} \mathrm{Ind}_H^G(W) &\longrightarrow \mathbb{K}G \otimes_{\mathbb{K}H} W \\ (w_1, \ldots, w_r) &\longmapsto \sum_{i=1}^r s_i \otimes w_i. \end{aligned}$$

Examples 5.0.9.

1. $\mathrm{Ind}_H^G(\mathbb{K}H) \cong \mathbb{K}G$.
2. $\mathrm{Ind}_H^G(\mathbb{K}) \cong \mathbb{K}[G/H]$.
3. $M^\lambda \cong \mathbb{C}[\mathfrak{S}_n/\mathfrak{S}_\lambda] = \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(1_\lambda)$, where $1_\lambda \cong \mathbb{C}$ is the trivial representation of $\mathfrak{S}_n$.

Take

$$\begin{aligned} \Phi : H &\longrightarrow \mathrm{GL}_n(\mathbb{K}) \\ h &\longmapsto (\Phi_{i,j}(h))_{i,j=1}^n, \end{aligned}$$

which corresponds to the $\mathbb{K}H$-module W with basis $\{w_1, \ldots, w_n\}$. Denote by Φ_G the matrix representation of $\mathrm{Ind}_H^G(W)$. We now describe it with respect to the basis $\{s_i \otimes w_j\}_{i,j}$. We have

$$\begin{aligned} g(s_i \otimes w_j) &= (gs_i) \otimes w_j \\ &= s_{\sigma_g(i)} \otimes h_{g,\sigma_g(i)} w_j \\ &= s_{\sigma_g(i)} \otimes \sum_{t=1}^n \Phi_{t,j}(h_{g,i}) w_t \\ &= \sum_{t=1}^n \Phi_{t,j}(h_{g,i})(s_{\sigma_g(i)} \otimes w_t) \\ &= \sum_{\mu=1}^r \sum_{t=1}^n \delta(s_\mu^{-1} g s_i) \Phi_{t,j}(h_{g,i})(s_\mu \otimes w_t), \end{aligned}$$

where

$$\delta(s_\mu^{-1} g s_i) = \begin{cases} 1 & \text{if } \sigma_g(i) = \mu \\ 0 & \text{if } \sigma_g(i) \neq \mu \end{cases}$$
$$= \begin{cases} 1 & \text{if } s_\mu^{-1} g s_i \in H \\ 0 & \text{if } s_\mu^{-1} g s_i \notin H. \end{cases}$$

We have $\Phi^G(g) = \Big(\Phi'(s_j^{-1} g s_i)\Big)_{i,j}$, where $\Phi'(s_j^{-1} g s_i) = \delta(s_j^{-1} g s_i)\Phi(s_j^{-1} g s_i)$.

Take now any function $\Psi : H \longrightarrow \mathbb{C}$, and denote

$$\dot{\Psi} : G \longrightarrow \mathbb{C}$$
$$g \longmapsto \begin{cases} \Psi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H. \end{cases}$$

Proposition 5.0.10. *Let χ^G be the character of Φ^G. Then*

$$\chi^G(g) = \sum_{i=1}^{r} \dot{\chi}(s_i^{-1} g s_i) = \frac{1}{|H|} \sum_{y \in G} \dot{\chi}(y^{-1} g y).$$

Proof. We have

$$\begin{aligned} \chi^G(g) &= \operatorname{tr} \Phi^G(g) \\ &= \sum_{i=1}^{r} \operatorname{tr} \Phi'(s_i^{-1} g s_i) \\ &= \sum_{i=1}^{r} \dot{\chi}(s_i^{-1} g s_i). \end{aligned}$$

Then

$$\begin{aligned} \sum_{y \in G} \dot{\chi}(y^{-1} g y) &= \sum_{i=1}^{r} \sum_{h \in H} \dot{\chi}(h^{-1} s_i^{-1} g s_i h) \\ &= \sum_{i=1}^{r} |H| \dot{\chi}(s_i^{-1} g s_i). \end{aligned}$$

□

Theorem 5.0.11 (Froebenius reciprocity)**.** *Let W be a $\mathbb{K}H$-module, and V be a $\mathbb{K}G$-module. Then*

$$\langle \chi_W^G, \chi_V \rangle_G = \langle \chi_W, \chi_{V|_H} \rangle_H.$$

Proof. We have :

$$\begin{aligned}
\langle \chi_W^G, \chi_V \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \chi_W^G(g) \overline{\chi_V(g)} \\
&= \frac{1}{|G||H|} \sum_{g,y \in G} \dot{\chi}_W(y^{-1}gy) \overline{\chi_V(g)} \\
&= \frac{1}{|G||H|} \sum_{g \in G} \dot{\chi}_W(g) \overline{\chi_V(ygy^{-1})} \\
&= \frac{1}{|H|} \sum_{g \in G} \dot{\chi}_W(g) \overline{\chi_V(g)} \\
&= \frac{1}{|H|} \sum_{g \in H} \chi_W(g) \overline{\chi_V(g)} \\
&= \langle \chi_W, \chi_{V|_H} \rangle_H.
\end{aligned}$$

□

Chapter 6

Character of the Specht modules

Remark 6.0.12. Let $\mu \vdash n$. We have

$$p_\mu = \sum_\lambda x^\lambda_\mu S_\lambda = \sum_\lambda \xi^\lambda_\mu m_\lambda.$$

If $\lambda = (\lambda_1, \ldots, \lambda_k)$, and $\mu = (\mu_1, \ldots, \mu_m)$, then $\chi_{M^\lambda}(\mu)$ is the coefficient of $m_\lambda = X_1^{\lambda_1} \cdots X_k^{\lambda_k}$ is the following polynomial :

$$\prod_{i=1}^m \underbrace{(X_1^{\mu_i} + \cdots + X_k^{\mu_i})}_{=p_{\mu_i}} = \prod_{i=1}^m p_{\mu_i} = p^\mu.$$

So $\chi_{M^\lambda}(\mu) = \xi^\lambda_\mu$.

Remark 6.0.13. We have

$$S_\lambda = \sum_\mu \frac{1}{z_\mu} x^\lambda_\mu p_\mu,$$
$$h_\lambda = \sum_\mu \frac{1}{z_\mu} \xi^\lambda_\mu p_\mu.$$

Let R_n be the free abelian group with basis $\{[S^\lambda]\}_{\lambda \vdash n}$. All the elements of R_n are uniquely written as

$$\sum_\lambda \alpha_\lambda [S^\lambda] = \left[\bigoplus_\lambda (S^\lambda)^{\oplus \alpha_\lambda}\right].$$

Take $R_0 = \mathbb{Z}$, and define

$$R = \bigoplus_{n \geq 0} R_n.$$

We want this group to be a graded ring. Define

$$\begin{aligned} R_n \times R_m &\longrightarrow R_{n+m} \\ ([V],[W]) &\longmapsto [\mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}}(V \otimes W)]. \end{aligned}$$

This is a well defined product. So R is a commutative graded ring (with unit). Define

$$\begin{aligned} R \times R &\longrightarrow \mathbb{Z} \\ ([S^\lambda],[S^\mu]) &\longmapsto \langle [S^\lambda],[S^\mu] \rangle = \delta_{\lambda,\mu}. \end{aligned}$$

If $[V] = \sum_\lambda n_\lambda [S^\lambda]$, and $[W] = \sum_\lambda m_\lambda [S^\lambda]$, then

$$\langle [V],[W] \rangle = \sum_\lambda n_\lambda m_\lambda.$$

However,

$$\begin{aligned} \langle [V],[W] \rangle &= \langle \chi_V, \chi_W \rangle \\ &= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi_V(\sigma) \overline{\chi_W(\sigma)} \\ &= \sum_\mu \frac{1}{z_\mu} \chi_V(\mu) \overline{\chi_W(\mu)} \\ &= \sum_\mu \frac{1}{z_\mu} \chi_V(\mu) \chi_W(\mu). \end{aligned}$$

Define now

$$\begin{aligned} \Phi : \Lambda &\longrightarrow R \\ h_\lambda &\longmapsto [M^\lambda]. \end{aligned}$$

Theorem 6.0.14. *The $\mathbb{Z}$-morphism Φ is*

1. *a homomorphism of rings,*
2. *an isomorphism,*
3. *an isometry,*
4. *such that $\Phi(S_\lambda) = [S^\lambda]$.*

Proof. 1. It is sufficient to show that $\Phi(h_\lambda) = \Phi(h_{\lambda_1} \cdots h_{\lambda_k}) = \Phi(h_{\lambda_1}) \cdots \Phi(h_{\lambda_k})$. We have

$$\Phi(h_{\lambda_i}) = [M^{\lambda_i}] = [1_{\mathfrak{S}_{\lambda_i}}].$$

So

$$\begin{aligned}\Phi(h_{\lambda_1})\cdots\Phi(h_{\lambda_k}) &= [1_{\mathfrak{S}_{\lambda_1}}]\cdots[1_{\mathfrak{S}_{\lambda_k}}]\\ &= [\operatorname{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} 1_{\mathfrak{S}_\lambda}]\\ &= [M^\lambda].\end{aligned}$$

2. From Young's rule, we have that $[M^\lambda] = [S^\lambda] + \sum_{\mu\rhd\lambda}\mathcal{K}_{\mu,\lambda}[S^\mu]$, and so, $\{[M^\lambda]\}_\lambda$ form a basis of R. So Φ is an isomorphism.

3. Define

$$\begin{aligned}\Psi : R &\longrightarrow \Lambda_{\mathbb{Q}} = \mathbb{Q}\otimes_{\mathbb{Z}}\Lambda\\ [M^\lambda] &\longmapsto \sum_\mu \frac{1}{z_\mu}\xi_\mu^\lambda p_\mu = \sum_\mu \frac{1}{z_\mu}\chi_{M^\lambda}(\mu)p_\mu.\end{aligned}$$

Because f the evaluation on the basis $\{[M^\lambda]\}_\lambda$, we have that $\Psi\circ\Phi$ is the inclusion map $\Lambda \hookrightarrow \Lambda_{\mathbb{Q}}$, and si Ψ is an inverse of Φ. Then,

$$\begin{aligned}\langle\Psi([V]),\Psi([W])\rangle &= \sum_{\mu,\lambda}\frac{1}{z_\mu z_\lambda}\chi_V(\mu)\chi_W(\lambda)\underbrace{\langle p_\mu, p_\lambda\rangle}_{=z_\mu\delta_{\lambda,\mu}}\\ &= \sum_\mu \frac{1}{z_\mu}\chi_V(\mu)\chi_W(\mu)\\ &= \langle[V],[W]\rangle,\end{aligned}$$

and so Φ is isometric, as Ψ is.

4. We know that $h_\lambda = S_\lambda + \sum_{\mu>\lambda}\alpha_{\mu,\lambda}S_\mu$, and that $[M^\lambda] = [S^\lambda] + \sum_{\mu\rhd\lambda}\mathcal{K}_{\mu,\lambda}[S^\mu]$. But $\Phi(h_\lambda) = [M^\lambda]$, and so

$$\Phi(S_\lambda) = [S^\lambda] + \sum_{\mu\rhd\lambda}\gamma_\mu[S^\mu].$$

However,

$$\begin{aligned}1 &= \langle S_\lambda, S_\lambda\rangle\\ &= \langle\Phi(S^\lambda),\Phi(S^\lambda)\rangle\\ &= 1 + \sum_{\mu\rhd\lambda}\gamma_\mu^2,\end{aligned}$$

and so $\gamma_\mu = 0$.

□

Corollary 6.0.15. *We have* $\chi_{S^\lambda}(\mu) = x_\mu^\lambda$.

Proof. We have

$$\begin{aligned}\sum_\mu \frac{1}{z_\mu} x_\mu^\lambda p_\mu &= S_\lambda \\ &= \Psi([S^\lambda]) \\ &= \sum_\mu \frac{1}{z_\mu} \chi_{S^\lambda}(\mu) p_\mu.\end{aligned}$$

□

Corollary 6.0.16. *We have*

$$h_\lambda = S_\lambda + \sum_{\mu \rhd \lambda} \mathcal{K}_{\mu,\lambda} S_\mu.$$

Proof. We know that $[M^\lambda] = [S^\lambda] + \sum_{\mu \rhd \lambda} \mathcal{K}_{\mu,\lambda}[S^\mu]$. Apply Ψ to get the result. □

Take λ a partition. Then we already know that

$$\dim S^\lambda = \sharp\{\text{standards tableaux of shape } \lambda\}.$$

We also know that $\chi_{S^\lambda}(\mu) = x_\mu^\lambda$, where

$$p_\mu(X_1, \ldots, X_k) = \sum_\lambda x_\mu^\lambda S_\lambda.$$

Moreover, $\chi_{S^\lambda}(\text{id}) = \chi_{S^\lambda}((1^n)) = \dim S^\lambda$. Let $l_i = \lambda_i + k - i$. Then x_μ^λ is the coefficient pf X^l in

$$p_\mu(X_1, \ldots, X_k) \prod_{i<j} (X_i - X_j)$$

(see exercise set 10). Recall that $p_\mu = p_{\mu_1} \cdots p_{\mu_r}$, and so

$$\begin{aligned}p_{(1^n)}(X_1, \ldots, X_k) &= p_1(X_1, \ldots, X_k)^n \\ &= (X_1 + \cdots + X_k)^n.\end{aligned}$$

We now take interest in the other part :

$$\begin{aligned}\prod_{i<j}(X_i - X_j) &= \begin{vmatrix} 1 & X_k & \cdots & X_k^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_1 & \cdots & X_1^{k-1} \end{vmatrix} \\ &= \begin{vmatrix} X_1^{k-1} & \cdots & X_k^{k-1} \\ \vdots & \ddots & \vdots \\ X_1 & \cdots & X_k \\ 1 & \cdots & 1 \end{vmatrix} \\ &= \sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) X_k^{\sigma(1)-1} \cdots X_1^{\sigma(k)-1}.\end{aligned}$$

On the other hand, we have that

$$(X_1 + \cdots X_k)^n = \sum_{r_1,\ldots,r_k} \frac{n!}{r_1! \cdots r_k!} X_1^{r_1} \cdots X_k^{r_k}.$$

So the coefficient of X^l in $p_\mu(X_1, \ldots, X_k) \prod_{i<j}(X_i - X_j)$ is

$$\begin{aligned}
&\sum_\sigma \operatorname{sgn}(\sigma) \frac{n!}{(l_1 - \sigma(k) + 1)! \cdots (l_k - \sigma(1) + 1)!} \\
&= \sum_\sigma \operatorname{sgn}(\sigma) \frac{n!}{\prod_{j=1}^k (l_j - \sigma(k-j+1) + 1)!} && \sigma \text{ st. } l_j - \sigma(k-j+1) + 1 \geq 0 \\
&= \frac{n!}{l_1! \cdots l_k!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) \prod_{j=1}^k l_j(l_j - 1) \cdots (l_j - \sigma(k-j+1) + 2) \\
&= \frac{n!}{l_1! \cdots l_k!} \begin{vmatrix} 1 & l_k & l_k(l_k - 1) & \cdots \\ 1 & l_{k-1} & l_{k-1}(l_{k-1} - 1) & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 1 & l_1 & l_1(l_1 - 1) & \cdots \end{vmatrix} \\
&= \frac{n!}{l_1! \cdots l_k!} \begin{vmatrix} 1 & l_k & l_k^2 & \cdots & l_k^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & l_1 & l_1^2 & \cdots & l_1^{k-1} \end{vmatrix} \\
&= \frac{n!}{l_1! \cdots l_k!} \prod_{i<j} (l_i - l_j) && \text{Vandermonde det.}
\end{aligned}$$

So finally :

$$\dim S^\lambda = \frac{n!}{l_1! \cdots l_k!} \prod_{i<j} (l_i - l_j).$$

6.1 The hook formula

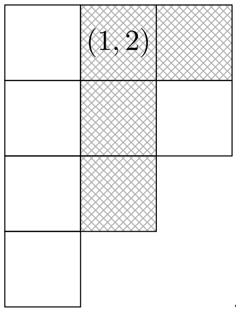
We start by assigning coordinates to the boxes in a Young diagram. For instance, if $\lambda = (3, 3, 2, 1) \vdash 9$, then we have

$[\lambda] =$

$(1,1)$	$(1,2)$	$(1,3)$
$(2,1)$	$(2,2)$	$(2,3)$
$(3,1)$	$(3,2)$	
$(4,1)$		

.

The *hook length* if the (i,j)-th box is the number of boxes that appear right, down, or on the box (i,j). For instance, in the previous diagram, the hook length of $(1,2)$ is $h(1,2) = 4$:



.

Theorem 6.1.1 (Hook formula)**.** *Let* $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ *be a partition. Then*

$$\dim S^\lambda = \frac{n!}{\prod_{i,j} h(i,j)}.$$

Proof. By induction on k, the number of rows in $[\lambda]$.

- If $k = 1$, then $\lambda = (n)$, and we already know that

$$\dim S^{(n)} = 1 = \frac{n!}{n!}.$$

- We know that

$$\begin{aligned}
\dim S^\lambda &= \frac{n!}{l_1! \cdots l_k!} \prod_{i<j} (l_i - l_j) \\
&= \frac{(n-\lambda_1)!}{l_2! \cdots l_k!} \left(\prod_{2 \le i < j \le k} (l_i - l_j) \right) \frac{(n - \lambda_1 + 1) \cdots n}{l_1!} \prod_{1 < j \le k} (l_1 - l_j) \\
&= \frac{(n-\lambda_1)!}{\prod_{i \ge 2,\, j} h(i,j)} (n - \lambda_1 + 1) \cdots n \prod_{1 < j \le k} \frac{l_1 - l_j}{l_1!} \\
&= \frac{n!}{\prod_{i \ge 2,\, j} h(i,j)} \prod_{1 < j \le k} \frac{l_1 - l_j}{l_1!}.
\end{aligned}$$

It remains to prove that the product of the hook lengths of the first row is

$$\frac{l_1!}{(l_1 - l_k) \cdots (l_1 - l_2)}.$$

Recall that $l_i = \lambda_i + k - i$. We have that

- $h(1,1) = \lambda_1 + k - 1 = l_1$,

– $h(1,2) = l_1 - 1$,
– ...
– $h(1,\lambda_k) = l_1 - \lambda_k + 1$,
– $h(1,\lambda_k + 1) = l_1 - \lambda_k - 1$, provided that $\lambda_{k-1} > \lambda_k$,
– ...

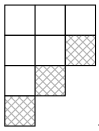
We conclude by this observation.

□

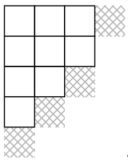
Example 6.1.2.

$$\dim S^{(2,1)} = \frac{3!}{3 \times 1 \times 1} = 2.$$

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition. We say that the box (i,j) is an *inner corner* of λ if, when we remove it from $[\lambda]$, we obtain the Yound diagram of another partition. For instance :



We denote by λ^- the set of partitions that are obtained by removing an inner corner of λ. Conversely, we say that a position $(i,j) \notin [\lambda]$ is an *outer corner* of λ if, when we add it to $[\lambda]$, we obtain the diagram of another partition. For instance :



We denote by λ^+ the set of partitons that are obtained by adding an outer corner of λ.

Lemma 6.1.3. *Let $f^\lambda = \dim S^\lambda$. Then $f^\lambda = \sum_{\mu \in \lambda^-} f^\mu$.*

Proof. Let T be a standard tableau of shape $\lambda \vdash n$. Necessarily, n appears in an inner corner of T, and we obtain another standard tableau by removing it. □

Theorem 6.1.4 (Branson rule)**.** *Take $\lambda \vdash n$. Then*

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(S^\lambda) = \bigoplus_{\mu \in \lambda^-} S^\mu,$$

$$\mathrm{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(S^\lambda) = \bigoplus_{\mu \in \lambda^+} S^\mu.$$

Proof. Let $\lambda \vdash n$, and $r_1 < \cdots < r_k$ the indexes of the rows of λ that contain an inner corner.

- We'll note by λ^i the partition obtained by removing the inner corner at row r_i. Remark that a row can't bear more that one inner corner.
- If T is any tableau of shape λ, and n appears in the inner corner at row r_i, then we'll denote by T^i the tableau obtained by removing the box containing n.
- Let $\{T\}$ be a tabloid with the property that n appears in a row containing an inner corner. We'll denote by $\{T^i\}$ the tabloid obtained by removing n.

Let G be a finite group, and V a $\mathbb{C}G$-module. Let $0 \neq W \subseteq V$ be a submodule. Then $V \cong W \oplus V/W$. Take a sequence

$$0 = V_0 \subseteq \cdots \subseteq V_k = S^\lambda$$

of $\mathbb{C}\mathfrak{S}_{n-1}$-modules with the property that $V_i/V_{i-1} \cong S^{\lambda^i}$ as $\mathbb{C}\mathfrak{S}_{n-1}$-modules. Then we are done. Denote by V^i the $\mathbb{C}$-vector space spanned by all the standard polytabloids e_T, where n appears in a row between r_1 and r_i. It is clear that $V_0 = 0$, $V_i \subseteq V_{i+1}$, and $V_k = S^\lambda$. Define

$$\begin{aligned}\Theta_i : M^\lambda &\longrightarrow M^{\lambda^i} \\ \{T\} &\longmapsto \begin{cases} \{T^i\} & \text{if } n \text{ appears in the row } r_i \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

It is a $\mathbb{C}\mathfrak{S}_{n-1}$-homomorphism, as "$n$ never moves". Take T a standard tableau. We have

$$\Theta_i(e_T) = \begin{cases} e_{T^i} & \text{if } n \text{ appears in row } r_i \\ 0 & \text{if } n \text{ appears in row } r_j < r_i.\end{cases}$$

Hence $\Theta_i(V_i) = \langle e_{T^i} \mid T^i \text{ standard of shape } \lambda^i\rangle = S^{\lambda^i}$, and $V_{i-1} \subseteq \ker \Theta_i$. So

$$0 = V_0 \subseteq V_1 \cap \ker \Theta_1 \subseteq V_1 \subseteq \cdots \subseteq V_k = S^\lambda,$$

and $\dim V_i/(V_i \cap \ker \Theta_i) =\sim S^{\lambda^i}$. On the other hand, $\sum \dim S^{\lambda^i} = \sum f^{\lambda^i} = f^\lambda$, and so $V_i = V_{i+1} \cap \Theta_{i+1}$, and $V_i/(V_i \cap \ker \Theta_i) = V_i/V_{i-1} = S^{\lambda^i}$. □

Index

— Symbols —

$C(T)$ 5
M^{λ} 6
$P(n)$ 1
R 37
$R(T)$ 5
R_n 37
S_{λ} 30
T_0 20
$[T]$ 18
Λ^k 26
Λ_n 25
Λ_n^k 25
α_a 29
$\mathcal{K}_{\lambda,\mu}$ 19
$\mathcal{T}_{\lambda,\mu}^0$ 19
$\mathcal{T}_{\lambda,\mu}$ 19
$\lambda(\sigma)$ 2
$\lambda \vdash n$ 1
λ^+ 43
λ^- 43
$\mu \trianglelefteq \lambda$ 3
$\uplus$ 16
$\{T'\} \trianglelefteq \{T\}$ 15
$\{T\}$ 6
a_H 9
b_H 9
e_T 9
e_n 27
h_n 27
k_T 9
m_{λ} 25
p_r 29

— C —

Column tabloid 18
Composition
    sequence 15
Compsition 14
Content 19

— D —

Dominance 3, 15

— G —

Garnir element 16, 17
Generalized Young tableau 19

— H —

Hook length 42

— I —

Inner corner 43

— K —

Kostka number 19

— O —

Outer corner 43

— P —

Part of a composition 14
Partition 1
Permutation module 6
Polytabloid 10
Power sum 29

— S —

Schur
    function 30
    polynomial 30
Semistandard tableau 19
Specht module 10
Standard

tableau 14
Symmetric polynomial 25

— T —

Tabloid 6
Type 2, 19

— Y —

Young
diagram 2
subgroup 2
tableau 2