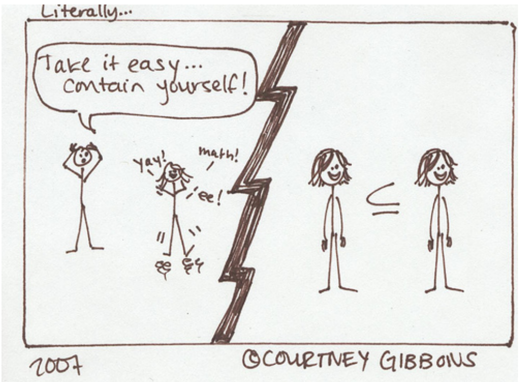
Set theory

Prof. J. Duparc
TeXed by Cédric Ho Thanh

Fall 2013



Contents

Introduction

Definition. This document is unofficial lecture notes from the set theory course, given by prof. J. Duparc during the fall semester 2013.

Corollary. *This document is provided as is, without warranty of any kind. Don't hesitate to spot mistakes so I can correct them.*

Chapter 1

Creating basic sets

1.1 First axioms of ZFC and elementary definitions

ZFC is a first order theory with equality = and signature $\{\in\}$.

Notations 1.1.1.

- $\exists x \in y \;\; \Phi$ stands for $\exists x \;\; (x \in y \wedge \Phi)$,
- $\forall x \in y \;\; \Phi$ stands for $\forall x \;\; (x \in y \wedge \Phi)$,

Axiom 1.1.2 (0. Set existence)**.**

$$\exists x \;\; x = x.$$

Axiom 1.1.3 (1. Extensionality)**.**

$$\forall x \forall y \;\; (\forall z \;\; (z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

Axiom 1.1.4 (2. Comprehension schema)**.**

$$\forall z \forall \vec{w} \exists y \forall x \;\; (x \in y \leftrightarrow x \in z \wedge \Phi),$$

where $\Phi = \Phi(x, z, \vec{w})$ is any formula with free variables among $x, z, w_1, \ldots, w_n$.

At this stage, we can define the unique set $\emptyset$ such that $\forall x \;\; x \notin \emptyset$. By axiom 1, $\exists x \;\; x = x$, and by axiom 2,

$$\exists \emptyset \forall z \;\; z \in \emptyset \iff z \in x \wedge z \neq z.$$

Axiom 1 states that $\emptyset$ is unique. The only set axiome 0, 1, 2 can determine is $\emptyset$. Consider $\mathcal{M}$ a model such that $|\mathcal{M}| = \{\emptyset\}$, $\in^{\mathcal{M}} = \{\emptyset\}$. Then

$$\mathcal{M} \models \mathrm{Ax}\,0, \mathrm{Ax}\,1, \mathrm{Ax}\,2.$$

Therefore, we cannot prove from axioms 0–2 the existence of a set that is different from $\emptyset$. This is a consequence of Godel's completeness theorem.

Theorem 1.1.5 (Russel)**.** *There is no universal set, i.e. a set of all sets.*

The theory Z is ZFC without the replacement schema and the axiom of choice. We prove something stronger :

Proposition 1.1.6. $\mathrm{Z} \vdash \neg\exists z \forall x \;\; x \in z$.

Proof. Assume that such a universal set z exists. By the comprehension axiom, $y = \{x \in z \mid x \notin x\} \in z$. However, $y \in y \leftrightarrow y \notin y$, a contradiction. □

Notation 1.1.7. $A \subseteq B$ stands for $\forall x \;\; x \in A \to x \in B$.

Axiom 1.1.8 (3. Pairing)**.**

$$\forall x \forall y \exists z \;\; x \in z \wedge y \in z.$$

Such a set z is *not* $\{x, y\}$. However, $\{x, y\} = \{k \in z \mid k = x \vee k = y\}$.

Definitions 1.1.9 (Pair, ordered pair)**.**

1. A *pair* is a set of the form $\{x, y\}$.
2. An *ordered pair* is a set of the form $\{\{x\}, \{x, y\}\}$, denoted by $\langle x, y\rangle$.

Proposition 1.1.10. *Given x, y, x', y', we have $\langle x, y\rangle = \langle x', y'\rangle$ iff $x = x'$ and $y = y'$.*

Proof. Exercise. □

Axiom 1.1.11 (4. Union)**.**

$$\forall a \exists b \forall x \forall y \;\; x \in y \wedge y \in a \to x \in b.$$

Such a set b is *not* $\bigcup a$. However $\bigcup a = \{x \in b \mid \exists y \;\; x \in y \wedge y \in a\}$. If $F \neq \emptyset$, we denote

$$\bigcap F = \left\{x \in \bigcup F \mid \forall y \in F \;\; x \in y\right\}.$$

Notations 1.1.12.

- $A \cup B$ stands for $\bigcup\{A, B\}$,
- $A \cap B$ stands for $\bigcap\{A, B\}$,
- $A \setminus B$ stands for $\{x \in A \mid x \notin B\}$.

Such a set y is *not* $\mathcal{P}(x)$. However, $\mathcal{P}(x) = \{k \in y \mid k \subseteq x\}$.

Axiom 1.1.13 (7. Replacement schema)**.**

$$\forall A \forall \vec{w} \;\; [\forall x \;\; (x \in A \to \exists! y \;\; \Phi) \to \exists Y \forall x \;\; (x \in A \to \exists y \;\; (y \in Y \wedge \Phi))],$$

where $\Phi = \Phi(x, y, A, \vec{w})$ is any formula with free variables among $x, y, A, w_1, \ldots, w_n$, and where $\exists! y \;\; \Phi$ abbreviates

$$\exists y \;\; \Phi(x, y, A, \vec{w}) \wedge (\forall z \;\; \Phi(x, z, A, \vec{w}) \to z = y).$$

By the comprehension schema, this gives $\{y \in Y \mid \exists x \in A \;\; \Phi(x, y, A, \vec{w})\}$ to be a set. The later set defines the "image" of the functional Φ.

Let A, B be two sets. By the replacement axiom used twice, we have that $\forall y \in B$,

$$\forall x \in A \exists! z \;\; z = \langle x, y \rangle \, .$$

By replacement and comprehension,

$$\mathrm{Prod}(A, y) = \{z \mid \exists x \in A \;\; z = \langle x, y \rangle\}$$

is a set. Then, $\{\mathrm{Prod}(A, y) \mid y \in B\}$ is also a set, and so

$$A \times B = \bigcup \{\mathrm{Prod}(A, y) \mid y \in B\}$$

is a set, called *cartesian product* of A and B.

Definition 1.1.14 (Binary relation)**.** A *binary relation* R is a set whose elements are ordered pairs. We note

$$\mathrm{dom}(R) = \left\{x \in \bigcup\bigcup R \mid \exists y \;\; \langle x, y \rangle \in R\right\},$$
$$\mathrm{ran}(R) = \left\{y \in \bigcup\bigcup R \mid \exists y \;\; \langle x, y \rangle \in R\right\}.$$

We have that $R \subseteq \mathrm{dom}(R) \times \mathrm{ran}(R)$. We define

$$R^{-1} = \{\langle y, x \rangle \in \mathrm{ran}(R) \times \mathrm{dom}(R) \mid \langle x, y \rangle \in R\}.$$

Definition 1.1.15 (Function, injection, surjection, bijection)**.** A *function* f is a relation that satisfies

$$\forall x \in \mathrm{dom}(f) \exists! y \in \mathrm{ran}(f) \;\; \langle x, y \rangle \in f.$$

We note $y = f(x)$. If $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) \subseteq B$, we note $f : A \longrightarrow B$. If $C \subseteq \mathrm{dom}(f)$, then

$$f[C] = \mathrm{ran}(f|_C) = \{f(x) \mid x \in C\}.$$

A function $f : A \longrightarrow B$ is :

- *injective* (or $1:1$) if f^{-1} is a function,
- *surjective* if $\mathrm{ran}(f) = B$,
- *bijective* if it is both injective and surjective.

1.2 Ordinal numbers

Definition 1.2.1 (Total ordering)**.** A *total ordering* is a pair $\langle A, R\rangle$, with R a binary relation, such that

1. R is transitive : $\forall x \forall y \forall z \;\; (xRy \wedge yRz) \rightarrow xRz$,
2. R is irreflexive : $\forall x \;\; \neg xRx$,
3. R is total on A : $\forall x \in A \forall y \in A \;\; x \neg y \rightarrow (xRy \vee yRx)$.

The relation R need not to be a subset of $A \times A$. If $B \subseteq A$, then $\langle B, R\rangle$ is a total ordering as well.

Definition 1.2.2 (Isomorphic sets with relations)**.** Let R, S be two binary relations, and A, B two sets. We say that $\langle A, R\rangle \cong \langle B, S\rangle$ if there exists a bijection $f : A \longrightarrow B$ such that

$$\forall x \in A \forall y \in A \;\; xRy \leftrightarrow f(x)Sf(y).$$

Definition 1.2.3 (Well ordering)**.** The pair $\langle A, R\rangle$ is a *well ordering* if

1. $\langle A, R\rangle$ is a well ordering,
2. for all subset $C \subseteq A$, $C \neg \emptyset$, then C admit a R-least element :

$$\forall C \subseteq A \;\; C \neg \emptyset \rightarrow (\exists x \in C \forall y \in C \;\; xRy).$$

For example, $\langle \mathbb{N}, <\rangle$ is a well ordering, but $\langle \mathbb{Z}, <\rangle$ isn't. If $\langle A, R\rangle$ is a total ordering, let us note

$$\text{Pred}(A, x, R) = \{y \in A \mid yRx\}.$$

Lemma 1.2.4. *If $\langle A, R\rangle$ is a well ordering, then $\forall x \in A$, we have*

$$\langle A, R\rangle \ncong \langle \text{Pred}(A, x, R), R\rangle .$$

Proof. Assume $f : A \longrightarrow \text{Pred}(A, x, R)$ is an isomorphism, and consider $Y = \{y \in A \mid f(y) \neq y\}$. We have that $Y \neq \emptyset$, for $x \in Y$. Take a, the R-least element of Y. We have that $f(a) \neq a$.

- If $f(a)Ra$, then since f is an isomorphism, we have $f(f(a))Rf(a)$, and so $f(f(a))Ra$. Moreover, $f(f(a)) \neq f(a)$, and so $f(a) \in Y$, which contradicts the minimality of a.
- If $aRf(a)$, then $f^{-1}(a)Ra$. We have that $f(f^{-1}(a)) = a \neq f^{-1}(a)$, and so $f^{-1}(a) \in Y$, which contradicts the mnimality of a once more.

□

Lemma 1.2.5. *Let $\langle A, R\rangle$ and $\langle B, S\rangle$ be two well orderings. If they are isomorphic, then such an isomorphism is unique.*

Proof. Assume $f, g : A \longrightarrow B$ are two different isomorphisms. Consider $Y = \{y \in A \mid f(y) \neq g(y)\}$. By hypothesis, $Y \neq \emptyset$. Let a be the R-least element of Y. Without loss of generality, $g(a) < f(a)$. Take $b = f^{-1}(g(a))$. We have that bRa since f is an isomorphism. Then $g(b) = f(b)$ by minimality of a, which implies $g(b) = g(a)$, a contradiction. □

Theorem 1.2.6. *Let $\langle A, R\rangle$ and $\langle B, S\rangle$ be two well orderings. Then one and only one of the following cases holds :*

1. $\langle A, R\rangle \cong \langle B, S\rangle$,
2. $\exists y \in B$ *such that* $\langle A, R\rangle \cong \langle \mathrm{Pred}(B, y, S), S\rangle$,
3. $\exists x \in A$ *such that* $\langle \mathrm{Pred}(A, x, R), R\rangle \cong \langle B, S\rangle$.

Proof. Let $f = \{\langle x, y\rangle \in A \times B \mid \langle \mathrm{Pred}(A, x, R), R\rangle \cong \langle \mathrm{Pred}(B, y, S), S\rangle\}$.

- We have that f is a function. Otherwise, $\exists v \in A, \exists w, w' \in B$ with $w \neq w'$ such that $\langle v, w\rangle, \langle v, w'\rangle \in f$. We have that $\langle \mathrm{Pred}(B, w, S), S\rangle \cong \langle \mathrm{Pred}(B, w', S), S\rangle$. Without loss of generality, wSw'. So $\mathrm{Pred}(B, w, S) \subsetneq \mathrm{Pred}(B, w', S)$, which is impossible.
- f is injective. Otherwise, if $f(v) = f(v') = w$ with $v \neq v'$ then we have $\langle \mathrm{Pred}(A, v, R), R\rangle \cong \langle \mathrm{Pred}(A, v', R), R\rangle$. Without loss of generality vRv', and so $\mathrm{Pred}(A, v, R) \subsetneq \mathrm{Pred}(A, v', R)$, which is impossible.
- f is an isomorphism from an initial segment of A to an initial segment of B. ???
- Those two initial segments cannot be both proper. ???

□

Definition 1.2.7 (Transitive set)**.** A set x is *transitive* if $\forall y \;\; y \in x \rightarrow y \subseteq x$.

Examples 1.2.8. The sets $\emptyset$, $\{\emptyset\}$, and $\{\emptyset, \{\emptyset\}\}$ are transitive. The set $\{\{\emptyset\}\}$ is not, since $\{\emptyset\} \in \{\{\emptyset\}\}$, but $\{\emptyset\} \nsubseteq \{\{\emptyset\}\}$.

Definition 1.2.9 (Ordinal number)**.** A set x is an *ordinal number* if

1. x is transitive,
2. $\langle x, \in_x\rangle$ is a well ordering, where $\in_x = \{\langle y, z\rangle \in x \times x \mid y \in z\}$.

Examples 1.2.10. The sets $\emptyset$, $\{\emptyset\}$, and $\{\emptyset, \{\emptyset\}\}$ are ordinal numbers. If $x = \{x\}$, then x is not an ordinal number, since $x \in_x x$, and so $\in_x$ is not a well ordering.

Notations 1.2.11.

- $x \cong \langle A, R\rangle$ stands for $\langle x, \in_x\rangle \cong \langle A, R\rangle$,
- if $y \in x$, then $\mathrm{Pred}(x, y)$ stands for $\mathrm{Pred}(x, y, \in_x)$.

Theorem 1.2.12. *1. If x is an ordinal number, and if $y \in x$, then y is also an ordinal number, and $y = \mathrm{Pred}(x, y)$.*

2. *If x and y are isomorphic ordinal numbers, then $x = y$.*

3. *If x and y are ordinal numbers, then one and only one of the following cases holds :*

    - $x = y$,
    - $x \in y$,
    - $y \in x$.

4. *If x, y, and z are ordinal numbers, then $x \in y \wedge y \in z \to x \in z$.*

5. *If C is a nonempty set of ordinal numbers, then it has a $\in$-least element.*

Proof. 1. $\langle y, \in_y \rangle$ is a well ordering. Since x is transitive, we have $b \in a \in y \in x \to b, a, y \in x$. Since x is a well ordering $b \in y$, and so y is itself transitive. Finally $a \in y \leftrightarrow a \in \text{Pred}(x, y)$, and so $y = \text{Pred}(x, y)$.

2. Assume $f : x \longrightarrow y$ is an isomorphism, and $x \Delta y \neq \emptyset$. Without loss of generality, $x \setminus y \neg \emptyset$. Take a the $\in$-least element of $x \setminus y$. Then $a = \text{Pred}(x, a) = \text{Pred}(y, f(a))$. Therefore $a = f(a)$, which is absurd.

3. From a previous result.

4. Remark that z is transitive.

5. Remark that $\in$ is a well ordering.

□

Theorem 1.2.13. $\neg \exists z \forall x \;\; x \textit{ ordinal} \to x \in z$.

Proof. Otherwise, if z is a set, by use of the comprehension axiom, the following class is a set :

$$\mathbf{ON} = \{x \in z \mid x \text{ ordinal}\}.$$

Then, **ON** is a transitive set, well ordered by $\in$. So **ON** is itself an ordinal, and $\mathbf{ON} \in \mathbf{ON}$, a contradiction with the well ordering of $\in$. □

Lemma 1.2.14. *If A is a transitive set of ordinals, then A is an ordinal.*

Proof. The set A is then transitive and well ordered by $\in$. □

Theorem 1.2.15. *If $\langle A, R \rangle$ is a well ordering, then there exists a unique ordinal C such that $\langle A, R \rangle \cong C$.*

Proof. • Unicity : if C' is another such ordinal, then $C \cong C'$, and so $C = C'$.

- Existance : define

$$B = \{a \in A \mid \exists x \;\; x \text{ ordinal} \wedge \text{Pred}(A, x, R) \cong x\}.$$

  Define f a function such that $\operatorname{dom} f = B$ and such that for all $a \in A$, $f(a)$ is the unique ordinal specified above. Denote $C = \operatorname{ran} f$. Then, C is an ordinal, as it is a transitive set of ordinals. Moreover, f is an isomorphism $\langle B, R\rangle \longrightarrow C$. If $B = A$, we're done. Otherwise, take a the R-least element of $A \setminus B$. Then $f|_{\text{Pred}(A,a,R)} : \langle \text{Pred}(A, a, R), R\rangle \longrightarrow f[\text{Pred}(A, a, R)]$ is an isomorphism. The later set is a transitive set of ordinals, and therefore an ordinal, which contradicts the definition of a.

□

Definition 1.2.16 (Type of a well ordering)**.** If $\langle A, R\rangle$ is a well ordering, define its *type* $\text{type}\langle A, R\rangle$ to be the unique ordinal C such that $\langle A, R\rangle \cong C$.

Definition 1.2.17 (Supremum, infimum)**.** Let X be a set of ordinals. Define its *supremum*

$$\sup X = \bigcup X.$$

If $X \neq \emptyset$, define its *infimum*

$$\inf X = \bigcap X.$$

We now use greek letters for ordinals. We denote

- $\alpha < \beta$ for $\alpha \in \beta$,
- $\alpha \leq \beta$ for $\alpha \in \beta \vee \alpha = \beta$,
- $s(\alpha) = \alpha \cup \{\alpha\}$, the *successor operation.*

Lemma 1.2.18. *1.* $\alpha \leq \beta \leftrightarrow \alpha \subseteq \beta$.

2. If X is a set of ordinals, then $\sup X$ *is the least ordinal greater or equal to any ordinal of X.*

3. If X is nonempty, then $\inf X$ *is the $\in$-least element of X.*

Proof. Clear. □

Lemma 1.2.19. *Let α, β be two ordinals.*

1. $s(\alpha)$ is an ordinal.

2. $\alpha < s(\alpha)$,

3. $\beta < s(\alpha) \rightarrow \beta \leq \alpha$.

Proof. Exercise. □

Definition 1.2.20 (Successor ordinal, limit ordinal)**.** An ordinal α is a *successor ordinal* if $\exists\beta$ another ordinal such that $\alpha = s(\beta)$. Otherwise, it is a *limit ordinal*.

We now use numbers to denote certain ordinals :

- $0 = \emptyset$,
- $n + 1 = s(n)$.

Definition 1.2.21 (Integer)**.** An ordinal α is an *integer* if

$$\forall \beta \leq \alpha \;\; \beta = 0 \vee \beta \text{ is a successor.}$$

Axiom 1.2.22 (5. Infinity)**.**

$$\exists x \;\; (\emptyset \in x \wedge \forall y \;\; (y \in x \rightarrow y \cup \{y\} \in x)).$$

Exercise 1.2.23. From axiom 5, we can construct the set ω of integers.

Definition 1.2.24 (Ordinal addition)**.** Let α and β be two ordinals. Define their *sum* :

$$\alpha + \beta = \text{type}\langle \alpha \times \{0\} \cup \beta \times \{1\}, R\rangle,$$

where

$$\begin{aligned} R = &\{\langle\langle a, 0\rangle, \langle b, 0\rangle\rangle \mid a < b, \; a, b \in \alpha\} \\ &\cup \{\langle\langle a, 0\rangle, \langle b, 1\rangle\rangle \mid a \in \alpha, \; b \in \beta\} \\ &\cup \{\langle\langle a, 1\rangle, \langle b, 1\rangle\rangle \mid a < b, \; a, b \in \beta\}. \end{aligned}$$

Theorem 1.2.25. *1.* $0 \in \omega$.

2. $\forall n \in \omega \;\; s(n) \in \omega$.

3. $\forall m, n \in \omega \; m \neq m \rightarrow s(n) \neq s(m)$.

4. (Induction) $\forall X \subseteq \omega \;\; (0 \in X \wedge \forall n \in X(s(n) \in X)) \rightarrow X = \omega$.

Proof. Easy. □

Lemma 1.2.26. *Let α, β and γ be ordinals.*

1. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

2. $\alpha + 0 = \alpha$.

3. $\alpha + 1 = s(\alpha)$.

4. $\alpha + s(\beta) = s(\alpha + \beta)$.

5. If β is limit, then $\alpha + \beta = \sup\{\alpha + \gamma \mid \gamma < \beta\}$.

Proof. Immediate from the definition. □

Warning : ordinal addition is not commutative ! It is associative though.

Definition 1.2.27 (Ordinal multiplication)**.** Let α and β be two ordinals. Define their *product*

$$\alpha \cdot \beta = \alpha\beta = \text{type}\langle \alpha \times \beta, R\rangle,$$

where R is the lexocographical order, i.e. $\langle a, b\langle R\langle a', b'\rangle$ if $a < a'$ or if $a = a'$ and $b < b'$.

Lemma 1.2.28. *Let α, β and γ be ordinals.*

1. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.
2. $\alpha 0 = 0\alpha = 0$.
3. $\alpha 1 = 1\alpha = \alpha$.
4. $\alpha s(\beta) = \alpha\beta + \alpha$.
5. *If β is limit, then $\alpha\beta = \sup\{\alpha\gamma \mid \gamma < \beta\}$.*
6. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

Proof. Immediate from the definition. □

Warning : ordinal multiplication is not commutative ! It is associative though.

Notations 1.2.29. Let A be an set. Denote

- A^n for the set of functions $f : n \longrightarrow A$,
- $A^{<\omega}$ for $\bigcup\{A^n \mid n < \omega\}$,
- $\langle x_0, \ldots, x_{n-1}\rangle$ for the function S such that $\text{dom}\, S = n$, and $S(k) = x_k$, $\forall k < n$. Warning : this notation is not consistant with the ordered pair. We will distinguish between those two when necessary (almost never).

Definition 1.2.30 (Class, proper class)**.** Let Φ be a first order set-theoretic formula. We call a collection of the form $\{x \mid \Phi(x)\}$ a *class*. A class is *proper* if it isn't a set.

By comprehenion, if C is a class and A a set such that $C \subseteq A$, then C is also a set. We define two important proper classes :

- $\mathbf{V} = \{x \mid x = x\}$, the class of all sets,
- $\mathbf{ON} = \{x \mid x \text{ ordinal}\}$, the class of all ordinals.

Theorem 1.2.31 (Transfinite induction on **ON**)**.** *If $C \subseteq \mathbf{ON}$ is nonempty, then it admits a least element.*

Proof. Take any $\alpha \in C$. Then $\alpha \cap C \subseteq \alpha$, and therefore is a set. If $\alpha \cap C = \emptyset$, then α is the least element of C. Otherwise, it is well ordered by $\in$, and therefore admit a least element, which is also the least element of C. □

Theorem 1.2.32 (Transfinite recursion on **ON**)**.** *If $F : \mathbf{V} \longrightarrow \mathbf{V}$ is a functional class, then there exists a unique class $G : \mathbf{ON} \longrightarrow \mathbf{V}$ such that $\forall \alpha \in \mathbf{ON}$*

$$G(\alpha) = F(G|_\alpha).$$

Proof.

- Unicity : Let G_1 and G_2 be two such functions. We prove that $G_1(\alpha) = G_2(\alpha)$, $\forall \alpha \in \mathbf{ON}$. Let α be the least ordinal such that $G_1(\alpha) \neq G_2(\alpha)$. We have

$$\begin{aligned} G_1(\alpha) &= F(G_1|_\alpha) \\ &= F(G_2|_\alpha) \\ &= G_2(\alpha), \end{aligned}$$

  a contradiction.

- Existence : We say that g is a δ-approximation of G if g is a function such that $\text{dom}(g) = \delta$, and $g(\alpha) = F(g|_\alpha)$, $\forall \alpha < \delta$.
  - We show by transfinite induction that if g is a δ-approximation of G, and if g' is a δ'-approximation of G, then $g|_{\delta \cap \delta'} = g'|_{\delta \cap \delta'}$. This is similar to the proof of the uniqueness of G.
  - We show by transfinite induction on δ that for each δ, there exists a (necessarily unique) δ-approximation. Again, the proof is similar to the previous ones.
  - We define G as $G(\alpha) = g(\alpha)$, for any $\delta > \alpha$, and g a δ-approximation.

□

We can now define thing using transfinite recursion, for instance the ordinal operations :

Definition 1.2.33 (Ordinal addition)**.** Let α and β be two ordinals. Define the *ordinal addition* by transfinite recursion :

1. $\alpha + 0 = \alpha$,
2. $\alpha + s(\beta) = s(\alpha + \beta)$,
3. $\alpha + \beta = \sup\{\alpha + \gamma \mid \gamma < \beta\}$, if β is limit.

Definition 1.2.34 (Ordinal multiplication)**.** Let α and β be two ordinals. Define the *ordinal multiplication* by transfinite recursion :

1. $\alpha * 0 = 0$,
2. $\alpha * s(\beta) = \alpha * \beta + \alpha$,
3. $\alpha * \beta = \sup\{\alpha * \gamma \mid \gamma < \beta\}$, if β is limit.

Definition 1.2.35 (Ordinal exponentiation)**.** Let α and β be two ordinals. Define the *ordinal exponentiation* by transfinite recursion :

1. $\alpha^0 = 1$,
2. $\alpha^{s(\beta)} = \alpha^\beta \alpha$,
3. $\alpha^\beta = \sup\{\alpha^\gamma \mid \gamma < \beta\}$, if β is limit.

Definition 1.2.36 (Predecessor)**.** Let α be an ordinal. We define its *predecessor* as :

1. $\mathrm{Pred}(\alpha) = \beta$, where $s(\beta) = \alpha$, if α is a sucessor ordinal,
2. $\mathrm{Pred}(\alpha) = \alpha$, if α is limit.

1.3 Cardinal numbers

Notation 1.3.1. Let A and B be two sets. We note

- $A \preceq B$ if there eists an injection $A \longrightarrow B$,
- $A \cong B$ if there exists a bijection $A \longrightarrow B$,
- $A \prec B$ if $A \preceq B$ and $A \ncong B$.

Theorem 1.3.2 (Cantor–Schröder–Bernstein)**.** *If $A \preceq B$ and $B \preceq A$, then $A \cong B$.*

Proof. Exercise. □

Definition 1.3.3 (Cardinal of a well ordered set)**.** Let A be a set. If A can be well ordered, we define its *cardinal* $|A| = \mathrm{Card}(A)$ as the least ordinal α such that $\alpha \cong A$, for some well ordering of A.

With the axiom of choice (AC), the cardinal of any set is well defined. Take $\alpha \in \mathbf{ON}$. Then $|\alpha| \leq \alpha$. Moreover, $|\alpha| = \alpha$, $\forall \alpha \leq \omega$.

Definition 1.3.4 (Cardinal number)**.** An ordinal α is a cardinal number if $|\alpha| = \alpha$. This is equivalent to $\beta \ncong \alpha$ as sets, $\forall \beta < \alpha$.

We use letters κ and λ for cardinals. Denote by **CARD** the class of all cardinals.

Lemma 1.3.5. *If $|\alpha| \leq \beta \leq \alpha$, then $|\beta| = |\alpha|$.*

Proof. We have that $\beta \subseteq \alpha$. Since $\alpha \cong |\alpha| \subseteq \beta$, we have $\alpha \leq \beta$. So $\alpha \cong \beta$ as sets, and $|\alpha| = |\beta|$. □

Lemma 1.3.6. *If $n \in \omega$, then*

1. $n \ncong n + 1$,
2. $\forall \alpha \;\; \alpha \cong n \rightarrow \alpha = n$.

Proof. Immediate. □

Corollary 1.3.7. *Each integer n is a cardinal. The ordinal ω is also a cardinal.*

Proof. Immediate. □

Definition 1.3.8 (Cardinal addition and multiplication)**.** We define the *cardinal addiction* and *multiplication* as

$$\kappa \oplus \lambda = |\kappa \amalg \lambda|,$$
$$\kappa \otimes \lambda = |\kappa \times \lambda|.$$

Remark that $\oplus$ and $\otimes$ are both commutative.

Lemma 1.3.9. *$\forall m, n \in \omega$, we have $m \oplus n = m + n$, $m \otimes n = mn$.*

Proof. Immediate. □

Lemma 1.3.10. *Every infinite cardinal is a limit ordinal.*

Proof. Otherwise, if $\kappa = s(\alpha)$ is an infinite cardinal, then α si infinite as well, and so $1 + \alpha = \alpha$. We have

$$\begin{aligned} \kappa &= |\kappa| \\ &= |s(\alpha)| \\ &= |\alpha + 1| \\ &= |1 + \alpha| \\ &= |\alpha|, \end{aligned}$$

a contradiction. □

Theorem 1.3.11. *If κ is an infinite cardinal, then $\kappa \otimes \kappa = \kappa$.*

Proof. By transfinite induction on κ.

- If $\kappa = \omega$, then $\omega \times \omega \cong \omega$, hence $\omega \otimes \omega = \omega$.
- Assume the induction hypothesis hold for all infinite cardinals below κ. We define $\lhd$, a well ordering on $\kappa \times \kappa$:

  $$\langle \alpha, \beta \rangle \lhd \langle \alpha', \beta' \rangle \iff \begin{cases} \max\{\alpha, \beta\} < \max\{\alpha', \beta'\}, \\ \max\{\alpha, \beta\} = \max\{\alpha', \beta'\} \text{ and } (\alpha, \beta) < (\alpha', \beta') \text{ in the lexicog. order.} \end{cases}$$

  Each $\langle \alpha, \beta \rangle \in \kappa \times \kappa$ has at most $|(\max\{\alpha, \beta\} + 1) \times (\max\{\alpha, \beta\} + 1)|$ predecessors. Let $\lambda = |\max\{\alpha, \beta\} + 1|$. By induction hypothesis,

  $$\begin{aligned} |(\max\{\alpha, \beta\} + 1) \times (\max\{\alpha, \beta\} + 1)| &= |\lambda \times \lambda| \\ &= \lambda \otimes \lambda \\ &= \lambda \\ &< \kappa. \end{aligned}$$

  Hence, type$\langle \kappa \times \kappa, \lhd \rangle$, and so $\kappa \otimes \kappa = \kappa$.

□

Corollary 1.3.12. *Let κ and λ be two cardinals.*

1. *If at least one of them is infinite, then $\kappa \oplus \lambda = \kappa \otimes \lambda = \max\{\kappa, \lambda\}$.*
2. *If κ is infinite, then $|\kappa^{<\omega}| = \kappa$.*

Proof. 1. Clear.

2. We have $|\kappa^n| = \kappa \otimes n = \kappa$, so

$$\begin{aligned} |\kappa^{<\omega}| &= \left| \bigcup_{n\in\omega} \kappa^n \right| \\ &= \omega \otimes \kappa \\ &= \kappa \end{aligned}$$

since $\kappa \geq \omega$.

□

Axiom (6. Power set)**.**

$$\forall x \exists y \forall z \ \ (\forall u \ \ (u \in z \to u \in x) \to z \in y).$$

By comprehension and extentionality axioms, we obtain

$$\mathcal{P}(x) = \{z \mid z \subseteq x\}$$

to be a set.

Theorem 1.3.13 (Cantor)**.** $\forall x \, x \prec \mathcal{P}(x)$.

Proof. Toward a contradiction, suppose that there exists a surjection $f : x \longrightarrow \mathcal{P}(x)$. Consider $S = \{y \in x \mid y \notin f(y)\}$. One has that $y \in S \leftrightarrow y \notin S$, a contradiction. □

By AC, there exists a cardinal above ω, namely $|\mathcal{P}(\omega)|$. However, we do not require AC to prove that :

Theorem 1.3.14. $\forall \alpha \in \mathbf{ON} \exists \kappa \in \mathbf{CARD} \ \ \alpha < \kappa$.

Proof. Assume $\alpha \geq \omega$, and let $W = \{R \in \mathcal{P}(\alpha \times \alpha) \mid R \text{ is a well ordering}\}$. We have that $W \neq \emptyset$, since α is already well ordered. Let $S = \{\text{type}\langle \alpha, R\rangle \mid R \in W\}$. Then $\sup(S)$ is a cardinal above α. Indeed :

- $\sup(S)$ is a cardinal. S is a set of ordinals, closed under successor operation. If $\sup(S)$ isn't a cardinal, then S is an ordinal of S, a contradiction.
- $\sup(S) > \alpha$, since $s(\alpha) \in S$.

□

Notation 1.3.15. Let α^+ be the least cardinal strictly above α.

Definition 1.3.16 (Successor cardinal, limit cardinal)**.** A cardinal κ is a *successor caridinal* if $\kappa = \alpha^+$, for some ordinal α. It as *limit* otherwise.

Definition 1.3.17 ($\aleph_\alpha$)**.** The cardinals $\aleph_\alpha = \omega_\alpha$ are defined by transfinite recursion

1. $\aleph_0 = \omega$,
2. $\aleph_{s(\alpha)} = \aleph_\alpha^+$,
3. $\aleph_\alpha = \sup\{\aleph_\gamma \mid \gamma < \alpha\}$, if α is limit.

Lemma 1.3.18. *1. $\aleph_\alpha$ is a cardinal, for each ordinal α.*

2. Each cardinal κ satisfies $\exists \alpha \in \mathbf{ON} \;\; \kappa = \aleph_\alpha$.

3. If $\alpha < \beta$, then $\aleph_\alpha < \aleph_\beta$.

4. $\aleph_\alpha$ is a limit cardinal iff α is a limit ordinal. Equivalentely, $\aleph_\alpha$ is a successor cardinal iff α isa successor ordinal.

Lemma 1.3.19. *With AC. If there is a surjective mapping $X \longrightarrow Y$, then $|X| \geq |Y|$.*

Proof. Let R be a well ordering of X. Then

$$\begin{aligned} g : Y &\longrightarrow X \\ y &\longmapsto \min_R f^{-1}\{y\} \end{aligned}$$

is well defined and injective. Hence, $|Y| \leq |X|$. □

Lemma 1.3.20. *With AC. If $\kappa \geq \aleph_0$, and $\{X_\alpha\}_{\alpha<\kappa}$ is a collection of sets such that $|X_\alpha| \leq \kappa$, $\forall \alpha \leq \kappa$, then*

$$\left| \bigcup_{\alpha<\kappa} X_\alpha \right| \leq \kappa.$$

Proof. For each $\alpha < \kappa$, we choose $f_\alpha : X_\alpha \longrightarrow \kappa$ an injection. We define

$$\begin{aligned} f : \bigcup_{\alpha<\kappa} X_\alpha &\longrightarrow \kappa \times \kappa \\ x &\longmapsto (\alpha, f_\alpha(x)) \qquad\qquad \alpha = \min_{x \in X_\beta} \beta. \end{aligned}$$

Clearly f is injective, hence

$$\begin{aligned} \left| \bigcup_{\alpha<\kappa} X_\alpha \right| &\leq |\kappa \times \kappa| \\ &= \kappa. \end{aligned}$$

□

Levy proved that if ZF is consistant

- there exists a model such that $\mathcal{P}(\omega)$ is a countable union of countable sets,
- there exists a model where $\aleph_1$ is a countable union of countable sets.

Theorem 1.3.21. *With AC. Let κ be an infinite cardinal, B a set such that $|B| \leq \kappa$, S a set of functions such that $|S| \leq \kappa$. Let A be the closure of B uner the functions of S. Then $|A| \leq \kappa$.*

Proof. Exercise. □

Definition 1.3.22 (Set exponentiation, cardinal exponentiation)**.** Let A and B be two sets. We define their *exponentiation* as :

$$^{B}A = A^{B} = \{f \in \mathcal{P}(B \times A) \mid f : A \longrightarrow B\}.$$

If λ and κ are two cardinals, we note $\kappa^{\lambda} = |^{\lambda}\kappa|$.

Example 1.3.23. $2^{\aleph_0} = |^{\omega}\{0,1\}|$.

Lemma 1.3.24. *Let λ and κ be two cardinals such that $2 \leq \kappa \leq \lambda$ and $\lambda \geq \aleph_0$. Then*

$$^{\lambda}\kappa \cong {}^{\lambda}2 \cong \mathcal{P}(\lambda).$$

Proof. Clearly, $^{\lambda}2 \preceq {}^{\lambda}\kappa$, and

$$\begin{aligned} ^{\lambda}\kappa &\preceq {}^{\lambda}\lambda \\ &\preceq \mathcal{P}(\lambda \times \lambda) \\ &\preceq \mathcal{P}(\lambda) \\ &\cong {}^{\lambda}2. \end{aligned}$$

□

Warning : $2^{\omega} = \sup\{2^{n} \mid n < \omega\} = \omega$ is an ordinal exponentiation, whereas $2^{\aleph_0} = |\mathcal{P}(\omega)| > \aleph_0$ is a cardinal exponentiation.

Lemma 1.3.25. *With AC. If κ, λ and σ are cardinals, then*

1. $\kappa^{\kappa \oplus \sigma} = \kappa^{\lambda} \otimes \kappa^{\sigma}$,
2. $(\kappa^{\lambda})^{\sigma} = \kappa^{\lambda \otimes \sigma}$.

Proof. Follows from basic set theory. □

Definition 1.3.26 (Continuum hypothesis)**.** The *continuum hypothesis* (CH) is $2^{\aleph_0} = \aleph_1$. The *generalised continuum hypothesis* (GCH) is $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Hilberts first problem (1900) was to know if CH holds. In 1941, Gödel showed that if ZF is consistant, there is a model $\mathcal{M}$ such that $\mathcal{M} \models$ ZF $+$GCH $+$AC. In 1963, Cohen showed that is ZF is consistant, there exists a model $\mathcal{N}$ such that $\mathcal{N} \models$ ZF $+\neg$CGH.

1.4 Cofinality

Definition 1.4.1 (Cofinal function)**.** A function between ordinals $f : \alpha \longrightarrow \beta$ is *cofinal* if $\operatorname{ran}(f)$ is strictly unbounded in β.

Definition 1.4.2 (Cofinality of an ordinal)**.** The cofinality of an ordinal β is the least ordinal α such that there exists a cofinal function $f : \alpha \longrightarrow \beta$. We denote by $\operatorname{cof}(\beta)$ its cofinality.

The cofinality of any successor ordinal is 1, and the mapping is

$$\begin{aligned} f : 1 &\longrightarrow \alpha + 1 \\ 0 &\longmapsto \alpha. \end{aligned}$$

The cofinality of ω is ω. The cofinality of α_ω is also ω, with the mapping

$$\begin{aligned} g : \omega &\longrightarrow \aleph_\omega \\ n &\longmapsto \aleph_n. \end{aligned}$$

Remark that $\operatorname{cof}(\beta) \leq \beta$, for all ordinal β.

Lemma 1.4.3. *Let β be an ordinal. There exists a cofinal function $f : \operatorname{cof}(\beta) \longrightarrow \beta$ that is strictly increasing.*

Proof. Let $g : \operatorname{cof}(\beta) \longrightarrow \beta$ be a cofinal function. Define

$$\begin{aligned} f : \operatorname{cof}(\beta) &\longrightarrow \beta \\ \gamma &\longmapsto \max\{g(\gamma), \sup\{f(\xi) + 1 \mid \xi < \gamma\}\}. \end{aligned}$$

□

Lemma 1.4.4. *If α is a limit ordinal, β any ordinal, and $f : \alpha \longrightarrow \beta$ a strictly increasing cofinal function. Then* $\operatorname{cof}(\alpha) = \operatorname{cof}(\beta)$.

Proof. Let $g : \operatorname{cof}(\alpha) \longrightarrow \alpha$ be a cofinal mapping.

- $\operatorname{cof}(\alpha) \leq \operatorname{cof}(\beta)$ since if $g_\beta : \operatorname{cof}(\beta) \longrightarrow \beta$ is cofinal and increasing, and if $h(\gamma) = \min\{\xi \mid g(\xi) < f(\xi)\}$, then $h : \operatorname{cof}(\beta) \longrightarrow \alpha$ is cofinal.
- $\operatorname{cof}(\beta) \leq \operatorname{cof}(\alpha)$ since we have a cofinal map $\operatorname{cof}(\alpha) \xrightarrow{g} \alpha \xrightarrow{f} \beta$.

□

Corollary 1.4.5. *For all ordinal α, we have* $\operatorname{cof}(\operatorname{cof}(\alpha)) = \operatorname{cof}(\alpha)$.

Definition 1.4.6 (Regular ordinal)**.** An ordinal β is *regular* if it is limit and $\beta = \operatorname{cof}(\beta)$.

Lemma 1.4.7. *Any regular ordinal is a cardinal.*

Proof. Suppose that β is a regular ordinal that is not a cardinal. Let $\alpha < \beta$ be an ordinal such that $\alpha \cong \beta$ as sets. Then every bijection $f : \alpha \longrightarrow \beta$ is cofinal, and so

$$\begin{aligned} \mathrm{cof}(\beta) &\leq \mathrm{cof}(\alpha) \\ &\leq \alpha \\ &< \beta, \end{aligned}$$

a contradition. □

Lemma 1.4.8. *1. ω is regular.*

2. With AC, κ^+ is regular, for all cardinal κ.

Proof. 1. Already done.

2. Assume that $f : \alpha \longrightarrow \kappa^+$ is cofinal, where $\alpha < \kappa^+$ is a ordinal. Then

$$\kappa^+ = \bigcup \{f(\alpha) \in \kappa^+ \mid \gamma < \alpha\},$$

and so $|\kappa^+| \leq \kappa$, as it is a union of κ-many sets of cardinality at most κ. A contradiction.

□

Lemma 1.4.9. *If α is a limit ordinal, then* $\mathrm{cof}(\aleph_\alpha) = \mathrm{cof}(\alpha)$.

Proof. Define the cofinal map

$$\begin{aligned} f : \alpha &\longrightarrow \aleph_\alpha = \bigcup_{\gamma<\alpha} \aleph_\gamma \\ \gamma &\longmapsto \aleph_\gamma. \end{aligned}$$

□

Definitions 1.4.10 (Weakly inaccessible and strongly inaccessible cardinals)**.** 1. A cardinal κ is *weakly inaccesible* if

(a) $\aleph_0 < \kappa$,

(b) κ is a limit cardinal,

(c) κ is regular.

2. A cardinal κ is *strongly inaccessible* if

(a) $\aleph_0 < \kappa$,

(b) κ is regular,

(c) $2^\lambda < \kappa$, for all cardinal $\lambda < \kappa$.

In ZF and in ZF + GCH, being strongly inaccessible implies being weakly inaccessible.

Lemma 1.4.11 (König). *If κ if an infinite cardinal, λ is a cardinal such that* $\mathrm{cof}(\kappa) \leq \lambda$, *then* $\kappa < \kappa^\lambda$.

Proof. Let $f : \lambda \longrightarrow \kappa$ be a cofinal map, and $g : \kappa \longrightarrow \kappa^\lambda$. We show that g cannot be surjective. Define

$$\begin{aligned} h : \lambda &\longrightarrow \kappa \\ \alpha &\longmapsto \min(\kappa \setminus \{g(\gamma)(\alpha) \in \kappa \mid \gamma < f(\alpha)\}). \end{aligned}$$

We verify that $h \notin \mathrm{ran}(g)$. Otherwise, let $\beta \in \kappa$ be an ordinal such that $g(\beta) = h$, and $\alpha \in \lambda$ such that $\beta < f(\alpha)$. Then $h(\alpha)$ is the least element of $\kappa \setminus \{g(\gamma)(\alpha) \in \kappa \mid \gamma < f(\alpha)\}$, a set that contains $g(\beta)(\alpha) = h(\alpha)$, a contradiction. □

1.5 Extension by definition

Assume that

1. $\mathcal{L}$ is a first order langage,
2. T is a $\mathcal{L}$-theory,
3. $\mathcal{L}' = \mathcal{L} \cup \{P\}$, where P is a n-ary relational symbol,
4. $T' = T \cup \{\forall x_1 \cdots \forall x_n \ \ (\Phi(x_1, \ldots, x_n) \leftrightarrow P(x_1, \ldots, x_n))\}$, where Φ is a $\mathcal{L}$-formula.

Then T' is an *extension by definition* of T. We can do a similar construction with a function symbol : assume that

1. $\mathcal{L}$ is a first order langage,
2. T is a $\mathcal{L}$-theory,
3. $\mathcal{L}' = \mathcal{L} \cup \{f\}$, where f is a n-ary function symbol,
4. $T' = T \cup \{\forall x_1 \cdots \forall x_n \forall y \ \ (\Phi(x_1, \ldots, x_n, y) \leftrightarrow y = f(x_1, \ldots, x_n))\}$, where Φ is a $\mathcal{L}$-formula such that
$$T \vdash \forall x_1 \cdots \forall x_n \exists! y \ \Phi(x_1, \ldots, x_n, y).$$

Then T' is an *extension by definition* of T.

Theorem 1.5.1. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two first order langages, such that $\mathcal{L} \subseteq \mathcal{L}'$. Let T by a $\mathcal{L}$-theory, and T' a $\mathcal{L}'$-theory that extends T by definition.*

1. *For all $\mathcal{L}'$-formula Ψ with free variables $x_1, \ldots, x_n$, there exists a $\mathcal{L}$-formula Φ, also with free variables $x_1, \ldots, x_n$, such that*
$$T' \vdash \forall x_1 \cdots \forall x_n \ \ (\Psi(x_1, \ldots, x_n) \leftrightarrow \Phi(x_1, \ldots, x_n)).$$
2. *T' is a* conservative extension *of T, i.e. if Φ is a $\mathcal{L}$-formula, then $T' \vdash \Phi$ implies $T \vdash \Phi$.*

Chapter 2

Creating models of ZFC

2.1 Well-founded sets

We work on ZF^-, the theory ZF without the foundation axiom.

Definition 2.1.1. We define the sets $R(\alpha)$ by transfinite recursion :

1. $R(0) = \emptyset$,
2. $R(\alpha + 1) = \mathcal{P}(R(\alpha))$,
3. $R(\alpha) = \bigcup_{\gamma<\alpha} R(\gamma)$, if α is a limit ordinal.

Definition 2.1.2 (The class of well-founded sets)**.** Define the class of well founded sets by

$$\mathbf{WF} = \bigcup\{R(\alpha) \mid \alpha \in \mathbf{ON}\}.$$

Elements of $\mathbf{WF}$ are called *well-founded sets*.

Lemma 2.1.3. *1. The set $R(\alpha)$ is transitive.*

2. $\forall\gamma \leq \alpha$, we have $R(\gamma) \subseteq R(\alpha)$.

Proof. 1. By transfinite induction :

- $R(0) = \emptyset$ is transitive.
- If $\alpha = \beta + 1$, since $R(\alpha) = \mathcal{P}(R(\beta))$, if $y \in x \in R(\alpha)$, then $y \in R(\beta)$. By induction hypothesis, we know that $y \subseteq R(\beta)$ as $R(\beta)$ is transitive. So $y \in \mathcal{P}(R(\beta)) = R(\alpha)$.
- Remark that any reunion of transitive sets is transitive.

2. By transfinite induction :

- If $\alpha = 0$, then nothing needs to be done.
- If $\alpha = \beta + 1$, then $R(\beta) \subseteq R(\alpha)$, hence $R(\gamma) \subseteq R(\alpha)$, $\forall\gamma \leq \beta < \alpha$.

- If α is limit, then the result follows by definition.

$\square$

Definition 2.1.4 (Rank)**.** For any $x \in \mathbf{WF}$, define its *rank* :

$$\mathrm{rk}(x) = \min\{\alpha \mid x \in R(\alpha + 1)\}.$$

Lemma 2.1.5. *Let α be an ordinal. Then*

$$R(\alpha) = \{x \in \mathbf{WF} \mid \mathrm{rk}(x) < \alpha\}.$$

Proof. We have

$$\begin{aligned} \mathrm{rk}(x) < \alpha &\iff \exists \beta < \alpha \;\; x \in R(\beta + 1) \\ &\iff x \in R(\alpha). \end{aligned}$$

$\square$

Lemma 2.1.6. *Take $y \in \mathbf{WF}$.*

1. $\forall x \in y$, we have $x \in \mathbf{WF}$ and $\mathrm{rk}(x) < \mathrm{rk}(y)$,

2. $\mathrm{rk}(y) = \sup\{\mathrm{rk}(x) + 1 \mid x \in y\}$.

Proof. 1. Let $\alpha = \mathrm{rk}(y)$. Then $y \in R(\alpha+1) = \mathcal{P}(R(\alpha))$. If $x \in y$, then $x \in R(\alpha)$ and so $\mathrm{rk}(x) < \alpha$.

2. If $\alpha = \sum\{\mathrm{rk}(x) + 1 \mid x \in y\}$, then the first point implies that $\alpha \leq \mathrm{rk}(y)$, and for each $x \in y$, $\mathrm{rk}(x) < \alpha$. Hence $y \in R(\alpha + 1)$, i.e. $\mathrm{rk}(y) \leq \alpha$.

$\square$

Lemma 2.1.7. *Let α be an ordinal.*

1. $\alpha \in \mathbf{WF}$, and $\mathrm{rk}(\alpha) = \alpha$.

2. $R(\alpha) \cap \mathbf{ON} = \alpha$.

Proof. 1. By transfinite induction on α :

- If $\alpha = 0$, then nothing needs to be done.
- If $\alpha = \beta + 1$, then $\beta \in R(\beta + 1)$, by hypothesis. Hence $R(\beta + 2) = R(\alpha + 1)$, and so $\alpha \in R(\alpha + 1)$. So $\alpha \in \mathbf{WF}$, and $\mathrm{rk}(\alpha) \leq \alpha$. Since $\mathrm{rk}(\beta) = \beta$ and $\beta \in \alpha$, we have $\mathrm{rk}(\alpha) \geq \beta + 1 = \alpha$. So $\mathrm{rk}(\alpha) = \alpha$.
- If α is limit, then $\alpha \subseteq R(\alpha)$ by induction hypothesis. Hence $\alpha \in R(\alpha + 1)$, and so $\mathrm{rk}(\alpha) \leq \alpha$, which shows that $\alpha \in \mathbf{WF}$. Moreover,

$$\begin{aligned} \mathrm{rk}(\alpha) &= \sup\{\mathrm{rk}(\gamma) + 1 \mid \gamma < \alpha\} \\ &= \sup\{\gamma \mid \gamma < \alpha\} \\ &= \alpha. \end{aligned}$$

2. Immediate from the previous point, knowing that $R(\alpha) = \{x \in \mathbf{WF} \mid \mathrm{rk}(x) < \alpha\}$. □

Lemma 2.1.8. *1. If $x \in \mathbf{WF}$, then $\bigcup x, \mathcal{P}(x), \{x\} \in \mathbf{WF}$, and have rank strictly less than $\mathrm{rk}(x)+\omega$.*

2. If $x, y \in \mathbf{WF}$, then $x \times y, x \cup y, x \cap y, \{x,y\}, \langle x,y\rangle, {}^y x \in \mathbf{WF}$, and have rank strictly less than $\max\{\mathrm{rk}(x), \mathrm{rk}(y)\} + \omega$.

Proof. Exercise. □

Lemma 2.1.9. *Let x be a set. Then $x \in \mathbf{WF} \leftrightarrow x \subseteq \mathbf{WF}$.*

Proof.
- $\rightarrow$: by transitivity of $\mathbf{WF}$.
- $\leftarrow$: Let $\alpha = \sup\{\mathrm{rk}(y)+1 \mid y \in x\}$. Then $x \subseteq R(\alpha)$, hence $x \in R(\alpha+1) \in \mathbf{WF}$.

□

Remark that $\forall n < \omega$ we have $|R(n)| < \omega$, and $|R(\omega)| = \omega$.

Definition 2.1.10 (Well-founded relation)**.** In ZF^- without the powerset axiom. A relation R is *well-founded* on A if

$$\forall X \subseteq A \ \ [X \neq \emptyset \rightarrow \exists y \in X \ (\neg \exists z \in X \ zRx)].$$

The element y is called *R-minimal* in X.

Lemma 2.1.11. *In ZF^-. If $A \in \mathbf{WF}$, then $\in$ is well-founded on A.*

Proof. Let $X \subseteq A$, $X \neq \emptyset$. Let $\alpha = \min\{\mathrm{rk}(y) \mid y \in X\}$. Take $y \in X$ such that $\mathrm{rk}(y) = \alpha$. Then y is $\in$-minimal in X. □

Lemma 2.1.12. *In ZF^-. If A is a transitive set, such that $\in$ is well-founded on A, then $A \in \mathbf{WF}$.*

Proof. By previous lemma, we show that $A \subseteq \mathbf{WF}$. By contradiction, assume that it is not the case. Let $X = A \setminus \mathbf{WF} \neq \emptyset$. Let y be $\in$-minimal in X. Either $y = \emptyset$, a contradiction as $\emptyset \in \mathbf{WF}$, or $y \neq \emptyset$, and take $z \in y$. Then $z \in X$, and y is not $\in$-minimal. A contradiction. □

Definition 2.1.13. In ZF^- without the powerset axiom.

1. Let A be a set. Define

$$\cup^0 A = A,$$
$$\cup^{n+1} A = \bigcup \cup^n A.$$

2. Define the *transitive closure* of A by

$$\mathrm{trcl}(A) = \bigcup \{\cup^n A \mid n \in \omega\}.$$

Lemma 2.1.14. *Let A be a set.*

1. $A \subseteq \text{trcl}(A)$.
2. $\text{trcl}(A)$ *is transitive.*
3. *If* $A \subseteq T$, *and* T *is transitive, then* $\text{trcl}(A) \subseteq T$.
4. *If* A *is transitive, then* $\text{trcl}(A) = A$.
5. *If* $x \in A$, *then* $\text{trcl}(x) \subseteq \text{trcl}(A)$.
6. $\text{trcl}(A) = A \cup \bigcup\{\text{trcl}(x) \mid x \in A\}$.

Proof. Exercise. □

Theorem 2.1.15. *In ZF⁻. For each set A, the following are equivalent :*

1. $A \in \mathbf{WF}$,
2. $\text{trcl}(A) \in \mathbf{WF}$,
3. $\in$ *is well founded on* $\text{trcl}(A)$.

Proof.

1. $\implies$ 2. If A is well founded, then by induction on n, we show that $\cup^n A$ is also well founded (because **WF** is closes under $\bigcup$). Hence $\text{trcl}(A)$ is well founded.

2. $\implies$ 3. By previous lemma.

3. $\implies$ 1. By previous lemma : $\text{trcl}(A)$ is well founded, transitive, and $\in$ is well founded on it. So $\text{trcl}(A) \in \mathbf{WF}$.

□

Axiom (8. Foundation)**.**

$$\forall x \ \ (\exists y \ \ y \in x) \rightarrow (\exists y \ \ y \in x \wedge \neg(\exists z \ \ z \in x \wedge z \in y)).$$

In other words, if $x \neq \emptyset$, then x admit a $\in$-minimal element.

Theorem 2.1.16. *In ZF⁻. The wollowing are equivalent :*

1. *The axiom of foundation.*
2. *For any set A,* $\in$ *is well-founded.*
3. $\mathbf{V} = \mathbf{WF}$.

Proof.

1. $\implies$ 2. Immediate.

2. $\implies$ 3. By previous lemma.

3. $\implies$ 1. Immediate by previous lemma.

□

From now on, denote $\mathbf{V}_\alpha = R(\alpha)$. So the axiom of fondation states that

$$\mathbf{V} = \bigcup_{\alpha \in \mathbf{ON}} \mathbf{V}_\alpha.$$

A class $\mathbf{R}$ is *well founded* on a class $\mathbf{A}$ if it statisfies the minimum condition, as in the definition of a well founded relation.

2.2 The Mostovski collapse

Definition 2.2.1. We say that a class $\mathbf{R}$ is *set-like* on a class $\mathbf{A}$ if $\forall x \in \mathbf{A}$, $\{y \in \mathbf{A} \mid y\mathbf{R}x\}$ is a set.

Definition 2.2.2. Let $\mathbf{R}$ be set-like on $\mathbf{A}$.

1. $\text{Pred}(\mathbf{A}, x, \mathbf{R}) = \{y \in \mathbf{A} \mid y\mathbf{R}\}$.

2.
$$\text{Pred}^0(\mathbf{A}, x, \mathbf{R}) = \text{Pred}(\mathbf{A}, x, \mathbf{R}),$$
$$\text{Pred}^{n+1}(\mathbf{A}, x, \mathbf{R}) = \bigcup\{\text{Pred}(\mathbf{A}, y, \mathbf{R}) \mid y \in \text{Pred}^n(\mathbf{A}, x, \mathbf{R})\}.$$

3.
$$\text{cl}(\mathbf{A}, x, \mathbf{R}) = \bigcup\{\text{Pred}^n(\mathbf{A}, x, \mathbf{R}) \mid n \in \omega\}.$$

Lemma 2.2.3. *If $\mathbf{A}$ is transitive, $\in$ be set-like on $\mathbf{A}$, and if $x \in \mathbf{A}$, then*

1. $\text{Pred}(\mathbf{A}, x, \in) = x$*,*

2. $\text{Pred}(\mathbf{A}, x, \in) = \cup^n x$*,*

3. $\text{cl}(\mathbf{A}, x, \in) = \text{trcl}(x)$

Proof. Exercise. □

Theorem 2.2.4. *In ZF⁻ without the powerset axiom. If $\mathbf{R}$ is well-founded and set-like on $\mathbf{A}$, then every nonempty subclass $\mathbf{X}$ of $\mathbf{A}$ admits a $\mathbf{R}$-minimal element.*

Proof. Take any element $x \in \mathbf{X}$. If x isn't $\mathbf{R}$-minimal in $\mathbf{X}$, then $\mathbf{X} \cap \text{cl}(\mathbf{A}, x, \mathbf{R}) \subseteq \mathbf{A}$ contains a $\mathbf{R}$-minimal element which is also $\mathbf{R}$-minimal for $\mathbf{X}$. □

Theorem 2.2.5. *In ZF⁻ without the powerset axiom. If $\mathbf{R}$ is a well founded and set-like on $\mathbf{A}$, and if $\mathbf{F} : \mathbf{A} \times \mathbf{V} \longrightarrow \mathbf{V}$, then there exists a unique mapping $\mathbf{G} : \mathbf{A} \longrightarrow \mathbf{V}$ such that $\forall x \in \mathbf{A}$ we have*

$$\mathbf{G}(x) = \mathbf{F}(x, \mathbf{G}|_{\text{Pred}(\mathbf{A},x,\mathbf{R})}).$$

Proof. This is a generalization of the proof of transfinite recursion for $\in$. □

Definition 2.2.6 (Rank)**.** In ZF^- without the powerset axiom. Let $\mathbf{R}$ be well founded and set-like on $\mathbf{A}$. Define

$$\mathrm{rk}(\mathbf{A}, x, \mathbf{R}) = \sup\{\mathrm{rk}(\mathbf{A}, y, \mathbf{R}) + 1 \mid y\mathbf{R}x,\ y \in \mathbf{A}\}.$$

Remark 2.2.7. In ZF^-. If $\mathbf{A}$ is a transitive class and $\in$ is well founded on $\mathbf{A}$, then $\mathbf{A} \subseteq \mathbf{WF}$, and $\forall x \in \mathbf{A}$ we have

$$\mathrm{rk}(\mathbf{A}, x, \in) = \mathrm{rk}(x).$$

Definition 2.2.8 (Mostovski collapse)**.** In ZF^- without the powerset axiom. If $\mathbf{R}$ is well founded and set-like on $\mathbf{A}$, then the *Mostovski collapse* is the functional $\mathbf{G}$ defined by

$$\mathbf{G}(x) = \{\mathbf{G}(y) \mid y\mathbf{R}x,\ y \in \mathbf{A}\}, \qquad \forall x \in \mathbf{A}.$$

The Mostovski collapse of $\mathbf{A}$ is the range of $\mathbf{A}$ by $\mathbf{G}$, which we denote by $\mathbf{M}$.

Lemma 2.2.9. *In ZF^- without the powerset axiom.*

1. $\forall x, y \in \mathbf{A},\ x\mathbf{R}y \to \mathbf{G}(x) \in \mathbf{G}(y)$.
2. $\mathbf{M}$ *is transitive.*
3. *In ZF^-.* $\mathbf{M} \subseteq \mathbf{WF}$.
4. *In ZF^-. If $x \in \mathbf{A}$, then* $\mathrm{rk}(\mathbf{A}, x, \mathbf{R}) = \mathrm{rk}(\mathbf{G}(x))$.

Proof.

1. Immediate from the definition.
2. Immediate from the definition.
3. By induction on x.
4. We have

$$\begin{aligned}
\mathrm{rk}(\mathbf{G}(x)) &= \sup\{\mathrm{rk}(y) + 1 \mid y \in \mathbf{G}(x)\} \\
&= \sup\{\mathrm{rk}(\mathbf{G}(y)) + 1 \mid y\mathbf{R}x,\ y \in \mathbf{A}\} \\
&= \sup\{\mathrm{rk}(\mathbf{A}, y, \mathbf{R}) + 1 \mid y\mathbf{R}x,\ y \in \mathbf{A}\} && \text{by induction} \\
&= \mathrm{rk}(\mathbf{A}, x, \mathbf{R}).
\end{aligned}$$

□

Definition 2.2.10 (Extensionnal relation)**.** In ZF^- without the powerset axiom. A relation $\mathbf{R}$ is *extensionnal* on $\mathbf{A}$ if $\forall x, y, z \in \mathbf{A}$ we have

$$(z\mathbf{R}x \leftrightarrow z\mathbf{R}y) \to x = y,$$

or equivalently if

$$x \neq y \to \mathrm{Pred}(\mathbf{A}, x, \mathbf{R}) \neq \mathrm{Pred}(\mathbf{A}, y, \mathbf{R}).$$

Lemma 2.2.11. *In ZF$^-$ without the powerset axiom. If* $\mathbf{A}$ *is a transitive class, then* $\in$ *is extensionnal on* $\mathbf{A}$.

Proof. If $\mathbf{A}$ is transitive, then $\forall x \in \mathbf{A}$ we have $x = \text{Pred}(\mathbf{A}, x, \in)$. □

Lemma 2.2.12. *In ZF$^-$ without the powerset axiom. If* $\mathbf{R}$ *is well founded, extensionnal and set-like on* $\mathbf{A}$*, then the Mostovski collapse* $\mathbf{G} : \mathbf{A} \longrightarrow \mathbf{M}$ *is an isomorphism.*

Proof. By definition, $\mathbf{G}$ is surjective. We show that it is injective as well. Suppose otherwise, and take x the $\mathbf{R}$-minimal element in $\{y \in \mathbf{A} \mid \exists z \in \mathbf{A}$ such that $y \neq z$, $\mathbf{G}(y) = \mathbf{G}(z)\}$. Fix $y \neq x$ such that $\mathbf{G}(x) = \mathbf{G}(y)$. Since $\mathbf{R}$ is extensionnal, we have two possibilities :

- For some $z \in \mathbf{A}$, we have $z\mathbf{R}x$ but $\neg z\mathbf{R}y$. Since $\mathbf{G}(x) = \mathbf{G}(y)$, there exists $w \in \mathbf{A}$ such that $w\mathbf{R}y$ and $\mathbf{G}(z) = \mathbf{G}(w)$, which contradicts the minimality of x.
- For some $z \in \mathbf{A}$, we have $\neg z\mathbf{R}x$ and $z\mathbf{R}y$. This case is symmetrical to the previous one.

So $\mathbf{G}$ is a bijection. The fact that it is an isomorphism follows from the definition. □

Theorem 2.2.13. *In ZF$^-$ without the powerset axiom. If* $\mathbf{R}$ *is well founded, extansionnal and set-like on* $\mathbf{A}$*, then there exists a transitive class* $\mathbf{M}$ *and an injective functional* $\mathbf{G} : \mathbf{A} \longrightarrow \mathbf{M}$ *such that* $\mathbf{G}$ *is an isomorphism between* $(\mathbf{A}, \mathbf{R})$ *and* $(\mathbf{M}, \in)$*. Moreover,* $\mathbf{G}$ *and* $\mathbf{M}$ *are unique.*

Proof. Existence comes from the previous lemmas. For the unicity, assume that $\mathbf{G}'$, $\mathbf{M}'$ satisfy the conditions of the theorem. By induction on x, we show that $\mathbf{G}(x) = \mathbf{G}'(x)$. □

Corollary 2.2.14. *In ZF$^-$ without the powerset axiom. If* $\in$ *is extensionnal on* $\mathbf{A}$*, then there exists a transitive class* $\mathbf{M}$ *and an isomorphism* $\mathbf{G} : (\mathbf{A}, \in) \longrightarrow (\mathbf{M}, \in)$.

2.3 Relativization and absoluteness

Definition 2.3.1 (Relativization of a formula)**.** If $\mathbf{M}$ is a class and Φ is a formula, then the *relativization* of Φ in $\mathbf{M}$, denoted by $\Phi^{\mathbf{M}}$ is defined by induction on the heigh of Φ :

- $(x = y)^{\mathbf{M}}$ is $x = y$,
- $(x \in y)^{\mathbf{M}}$ is $x \in y$,
- $(\Psi \wedge \Theta)^{\mathbf{M}}$ is $\Psi^{\mathbf{M}} \wedge \Theta^{\mathbf{M}}$,
- $(\neg\Phi)^{\mathbf{M}}$ is $\neg\Phi^{\mathbf{M}}$,
- $(\exists x \;\; \Phi)^{\mathbf{M}}$ is $\exists x \;\; (x \in \mathbf{M} \wedge \Phi^{\mathbf{M}})$.

Remark 2.3.2. We have

- $(\Psi \vee \Theta)^{\mathbf{M}}$ is $\Psi^{\mathbf{M}} \vee \Theta^{\mathbf{M}}$,
- $(\Psi \rightarrow \Theta)^{\mathbf{M}}$ is $\Psi^{\mathbf{M}} \rightarrow \Theta^{\mathbf{M}}$,

- $(\Psi \leftrightarrow \Theta)^{\mathbf{M}}$ is $\Psi^{\mathbf{M}} \leftrightarrow \Theta^{\mathbf{M}}$,
- $(\forall x \ \ \Phi)^{\mathbf{M}}$ is $\forall x \ \ (x \in \mathbf{M} \to \Phi^{\mathbf{M}})$.

Definition 2.3.3. Let $\mathbf{M}$ be a class.

1. We say that a formula Φ *holds* in $\mathbf{M}$ if $\Phi^{\mathbf{M}}$ is true.
2. We say that a theory T *holds* in $\mathbf{M}$ if $\Phi^{\mathbf{M}}$ is true, $\forall \Phi \in T$.

Lemma 2.3.4. *Let S and T be two $\mathcal{L}$-theries, and $\mathbf{M}$ be a class. If*

$$T \vdash (\mathbf{M} \neq \emptyset) \wedge (\mathbf{M} \textit{ is a model of } S),$$

then $\mathrm{Cons}(T) \to \mathrm{Cons}(S)$.

Proof. If S where inconsistant, then for some formula Φ, we would have $S \vdash \Phi \wedge \neg\Phi$. Hence, $\Phi^{\mathbf{M}} \wedge \neg\Phi^{\mathbf{M}}$ would be provable in T. □

Lemma 2.3.5. *If $\mathbf{M}$ is transitive, then the axiom of extensionality holds on $\mathbf{M}$.*

Proof. The relation $\in$ is extensionnal on every transitive class. □

Lemma 2.3.6. *If for all formula $\Phi(x, y, \overrightarrow{w})$ we have*

$$\forall y \forall \overrightarrow{w} \in \mathbf{M} \ \ \{x \in z \mid \Phi^{\mathbf{M}}(x, z, \overrightarrow{w})\} \in \mathbf{M},$$

then the comprehension axiom holds in $\mathbf{M}$.

Proof. Immediate. □

Corollary 2.3.7. *If $\forall x \in \mathbf{M}$, $\mathcal{P}(x) \subseteq \mathbf{M}$, then the comprehension axiom holds in $\mathbf{M}$.*

Lemma 2.3.8. *If $\mathbf{M}$ is transitive, then the power st axiom holds on $\mathbf{M}$ iff*

$$\forall x \in \mathbf{M} \exists y \in \mathbf{M} \ \ (\mathcal{P}(x) \cap \mathbf{M}) = y.$$

Proof. We have :

$$\begin{aligned}(\text{Power set ax.})^{\mathbf{M}} &\equiv \forall x \in \mathbf{M} \exists y \in \mathbf{M} \forall z \in \mathbf{M} \ \ (z \subseteq x \to z \in y)^{\mathbf{M}} \\ &\equiv \forall x \in \mathbf{M} \exists y \in \mathbf{M} \forall z \in \mathbf{M} \ \ (z \cap \mathbf{M} \subseteq x \to z \in y).\end{aligned}$$

We need to show that

$$\forall x \in \mathbf{M} \exists y \in \mathbf{M} \forall z \in \mathbf{M} \ \ (z \cap \mathbf{M} \subseteq x \to z \in y) \equiv \forall x \in \mathbf{M} \exists y \in \mathbf{M} \ \ (\mathcal{P}(x) \cap \mathbf{M}) \subseteq y.$$

Notice that since $\mathbf{M}$ is transitive, we have $z \in \mathbf{M} \to z \cap \mathbf{M} = z$.

$\to$ If $z \cap \mathbf{M} \subseteq x$, then $z \subseteq x$, and so $z \in \mathcal{P}(x)$. We then have that $z \in \mathcal{P}(x) \cap \mathbf{M}$, and so $z \in y$.

$\leftarrow$ Let $a \in \mathcal{P}(x) \cap \mathbf{M}$. We have $a \subseteq x$, hence $a \cap \mathbf{M} = a \subseteq x$, and so $a \in y$.

□

Lemma 2.3.9. *1. If $\forall x, y \in \mathbf{M} \exists z \in \mathbf{M}\ (x \in z \wedge y \in z)$, then the pairing axiom holds in $\mathbf{M}$.*

2. If $\forall x \in \mathbf{M} \exists z \in \mathbf{M}\ \bigcup x \subseteq z$, then the union axiom holds in $\mathbf{M}$.

3. If $\mathbf{M} \subseteq \mathbf{WF}$, then the axiom of foundation holds in $\mathbf{M}$.

Proof. Immediate. □

Definition 2.3.10 (Absoluteness)**.** Let Φ be a formula with free variables among $x_1, \dots, x_n$.

1. If $\mathbf{M} \subseteq \mathbf{N}$, we say that Φ is *absolute* for $\mathbf{M}$ and $\mathbf{N}$ if

$$\forall \overrightarrow{x} \in \mathbf{M}\ \ (\Phi^{\mathbf{M}}(\overrightarrow{x}) \leftrightarrow \Phi^{\mathbf{N}}(\overrightarrow{x})).$$

2. If $\mathbf{M}$ is a class, we say that Φ is *absolute* for $\mathbf{M}$ if it is absolute for $\mathbf{M}$ and $\mathbf{V}$, i.e.

$$\forall \overrightarrow{x} \in \mathbf{M}\ \ (\Phi(\overrightarrow{x}) \leftrightarrow \Phi^{\mathbf{M}}(\overrightarrow{x})),$$

as $\Phi^{\mathbf{V}} = \Phi$.

Remark 2.3.11. If $\mathbf{M} \subseteq \mathbf{N}$, and if Φ is absolute for $\mathbf{M}$ and absolute for $\mathbf{N}$, then it is absolute for $\mathbf{M}$ and $\mathbf{N}$.

Lemma 2.3.12. *If $\mathbf{M} \subseteq \mathbf{N}$, and if Φ and Ψ are absolute for $\mathbf{M}$ and $\mathbf{N}$, then $\neg\Phi$ and $\Phi \wedge \Psi$ are absolute for $\mathbf{M}$ and $\mathbf{N}$. In other words, the set of absolute formulas for $\mathbf{M}$ and $\mathbf{N}$ is closed under logical connectors.*

Corollary 2.3.13. *If Φ is a formula without quantifiers, then Φ is absolute for any class $\mathbf{M}$.*

Lemma 2.3.14. *If $\mathbf{M} \subseteq \mathbf{N}$ are both transitive classes, $y \in \mathbf{M}$, and Φ is absolute for $\mathbf{M}$ and $\mathbf{N}$, then $\exists x \in y\ \ \Phi$ and $\forall x \in y\ \ \Phi$ are absolute for $\mathbf{M}$ and $\mathbf{N}$. In other words, the set of absolute formulas for $\mathbf{M}$ and $\mathbf{N}$ is closed under quantifiers bounded by an element of $\mathbf{M}$.*

Proof. We have

$$\begin{aligned}
(\exists x \in y\ \ \Phi(x,y))^{\mathbf{M}} &\equiv (\exists x\ \ x \in y \wedge \Phi(x,y))^{\mathbf{M}} \\
&\equiv \exists x \in \mathbf{M}\ \ x \in y \wedge \Phi^{\mathbf{M}}(x,y) \\
&\equiv \exists x\ \ x \in y \wedge \Phi^{\mathbf{M}}(x,y) && \mathbf{M} \text{ is transitive} \\
&\equiv \exists x\ \ x \in y \wedge \Phi^{\mathbf{N}}(x,y) \\
&\equiv (\exists x \in y\ \ \Phi(x,y))^{\mathbf{N}}.
\end{aligned}$$

□

Definition 2.3.15 (Δ_0^0 formula)**.** The Δ_0^0 formulas are inductively constructed with the following rules :

1. $(x = y), (x \in y) \in \Delta_0^0$,
2. if $\Phi, \Psi \in \Delta_0^0$, then $(\neg\Phi), (\Phi \wedge \Psi) \in \Delta_0^0$,
3. if $\Phi \in \Delta_0^0$, then $(\exists x \ \ (x \in y \wedge \Phi)) \in \Delta_0^0$.

Corollary 2.3.16. *If* $\mathbf{M}$ *is transitive and* $\Phi \in \Delta_0^0$*, then* Φ *is absolute for* $\mathbf{M}$.

Notice that if

1. T is a $\mathcal{L}$-theory,
2. $\mathbf{M} \subseteq \mathbf{N}$,
3. $\mathbf{M} \models T$,
4. $T \vdash \forall \overrightarrow{x} \ \ (\Phi \leftrightarrow \Psi)$,

then Φ is absolute for $\mathbf{M}$ and $\mathbf{N}$ iff Ψ is.

Definition 2.3.17 (Absolute function)**.** If $\mathbf{M} \subseteq \mathbf{N}$ and $\mathbf{F}$ is a function, then we say that it is *absolute* if $\mathbf{F}(\overrightarrow{x} = y$ is an absolute formula for $\mathbf{M}$ and $\mathbf{N}$.

Theorem 2.3.18. *The following formulas are equivalent to a* Δ_0^0 *formula :*

- $x = y$,
- $x \in y$,
- $x \subseteq y$,
- $\{x\}$,
- $\{x, y\}$,
- $\langle x, y \rangle$,
- $\emptyset$,
- $x \cup y$,
- $x \cap y$,
- $x \setminus y$,
- $s(x)$,
- x *is transitive,*
- $\bigcup x$,
- $\bigcap x$*, with the convention that* $\bigcap \emptyset = \emptyset$.

Lemma 2.3.19. *Absolute notions are closed under composition, i.e. if* $\mathbf{M} \subseteq \mathbf{N}$, Φ, $\mathbf{F}$ *and* $\mathbf{G}_i$ *are absolute for* $\mathbf{M}$ *and* $\mathbf{N}$, *then so are*

- $\Phi(\mathbf{G}_1(\overrightarrow{x}, \ldots, \mathbf{G}_n(\overrightarrow{x}))$,
- $\mathbf{F}(\mathbf{G}_1(\overrightarrow{x}, \ldots, \mathbf{G}_n(\overrightarrow{x}))$.

Theorem 2.3.20. *The following relations and functions are absolute for each transitive model of* ZF^- *without the powerset axiom :*

- z *is an ordered pair,*
- $A \times B$,
- R *is a relation,*
- $\mathrm{dom}(R)$,
- $\mathrm{ran}(R)$,
- f *is a function,*
- f *is injective,*
- $\mathrm{rk}(x)$.

Theorem 2.3.21. *In* ZF^- *without the powerset axiom. All the axioms of ZF hold true inside* $\mathbf{WF}$.

Corollary 2.3.22. *We have*

1. $\mathrm{Cons}(ZF^-) \leftrightarrow \mathrm{Cons}(ZF)$,
2. $\mathrm{Cons}(ZFC^-) \leftrightarrow \mathrm{Cons}(ZFC)$.

Theorem 2.3.23. *The following functions and relations are defined inside* ZF^- *without the powerset axiom by formulas that are equivalent to* Δ_0^0 *formulas, and therefore are absolute for any transitive model :*

- x *is an ordinal,*
- x *is a limit ordinal,*
- x *is a successor ordinal,*
- x *is a finite ordinal,*
- ω.

Lemma 2.3.24. *If* $\mathbf{M}$ *is a transitive model of* ZF^- *without the powerset axiom, then each finite subset of* $\mathbf{M}$ *is in* $\mathbf{M}$.

Proof. By induction on the cardinality of x, a finitesubset of $\mathbf{M}$:

- If $|x| = 0$, then $x = \emptyset$, and so $x \in \mathbf{M}$.
- If $|x| = k+1$, then take any element $e \in x$, and consider $x \setminus \{e\}$. The later set is absolute and have cardinality k. We have that $\{e\}, x \setminus \{e\} \in \mathbf{M}$, and so $x = \{e\} \cup (x \setminus \{e\}) \in \mathbf{M}$.

□

Theorem 2.3.25. *The following are absolute for transitive models of ZF^- without the powerset axiom :*

- *x is finite,*
- *A^n,*
- *$A^{<\omega}$,*
- *R well orders A,*
- type(A, R),
- $\alpha + 1$,
- $\alpha \dot{-} 1$,
- $\alpha + \beta$,
- $\alpha\beta$.

2.4 Definability

Definition 2.4.1. For $n \in \omega$, $i, j < n$, define

1. $$\begin{aligned} \mathrm{Proj}(\mathbf{A}, R, n) &= \{s \in \mathbf{A}^n \mid \exists t \in R \text{ st } t|_n = s\}, \\ \mathrm{Diag}_\in(\mathbf{A}, n, i, j) &= \{s \in \mathbf{A}^n \mid s(i) \in s(j)\}, \\ \mathrm{Diag}_=(\mathbf{A}, n, i, j) &= \{s \in \mathbf{A}^n \mid s(i) = s(j)\}, \end{aligned}$$

2. by recursion on $k \in \omega$, define

$$\begin{aligned} D'_f(0, \mathbf{A}, n) &= \{\mathrm{Diag}_\in(\mathbf{A}, n, i, j) \mid i, j < n\} \\ &\quad \cup \{\mathrm{Diag}_=(\mathbf{A}, n, i, j) \mid i, j < n\}, \\ D'_f(k+1, \mathbf{A}, n) &= D'_f(k, \mathbf{A}, n) \\ &\quad \cup \{\mathbf{A}^n \setminus R \mid R \in D'_f(k, \mathbf{A}, n)\} && \text{(negation)} \\ &\quad \cup \{R \cap S \mid R, S \in D'_f(k, \mathbf{A}, n)\} && \text{(conjunction)} \\ &\quad \cup \{\mathrm{Proj}(\mathbf{A}, R, n) \mid R \in D'_f(k, \mathbf{A}, n)\}. && \text{(exists. quant.)} \end{aligned}$$

3.
$$D_f(\mathbf{A}, n) = \bigcup_{k\in\omega} D'_f(k, \mathbf{A}, n).$$

Lemma 2.4.2. *If $R, s \in D_f(\mathbf{A}, n)$, then*

1. $\mathbf{A}^n \setminus R \in D_f(\mathbf{A}, n)$,
2. $R \cap S \in D_f(\mathbf{A}, n)$,
3. $\mathrm{Proj}(\mathbf{A}, R, n) \in D_f(\mathbf{A}, n)$, *if* $R \in D_f \in (\mathbf{A}, n+1)$.

Lemma 2.4.3. *If $\Phi(x_0, \ldots, x_{n-1})$ is a formula, then for all $\mathbf{A}$,*

$$\{s \in \mathbf{A}^n \mid \Phi^{\mathbf{A}}(s(0), \ldots, s(n))\} \in D_f(\mathbf{A}, n).$$

Proof. Easy induction on $\mathrm{ht}(\Phi)$. □

Definition 2.4.4. By recursion on $n, m \in \omega$, we define $En(m, bfAA, n)$ by

- if $m = 2^i 3^j$, $i < j < n$, then $En(m, \mathbf{A}, n) = \mathrm{Diag}_{\in}(\mathbf{A}, n, i, j)$,
- if $m = 2^i 3^j 5$, $i < j < n$, then $En(m, \mathbf{A}, n) = \mathrm{Diag}_{=}(\mathbf{A}, n, i, j)$,
- if $m = 2^i 3^j 5^2$, $i < j < n$, then $En(m, \mathbf{A}, n) = A^n \setminus En(i, \mathbf{A}, n)$,
- if $m = 2^i 3^j 5^3$, $i < j < n$, then $En(m, \mathbf{A}, n) = En(i, \mathbf{A}, n) \cap En(j, \mathbf{A}, n)$,
- if $m = 2^i 3^j 5^4$, $i < j < n$, then $En(m, \mathbf{A}, n) = \mathrm{Proj}(\mathbf{A}, En(i, \mathbf{A}, n+1), n)$,
- $En(m, \mathbf{A}, n) = \emptyset$ otherwise.

Lemma 2.4.5. *For all $\mathbf{A}$ and all $n \in \omega$, we have*

$$D_f(\mathbf{A}, n) = \{E_n(m, \mathbf{A}, n) \mid n \in \omega\}.$$

Corollary 2.4.6. *We have $|D_f(\mathbf{A}, n)| \leq \omega$.*

Lemma 2.4.7. *The functions D_f and En are absolute for transitive models of ZF^- without the powerset axiom.*

Definition 2.4.8. Define

$$\mathcal{D}(\mathbf{A}) = \{X \subseteq \mathbf{A} \mid \exists n \in \omega,\ \exists s \in \mathbf{A}^n,\ \exists R \in D_f(\mathbf{A}, n+1) \text{ st } X = \{x \in A \mid s\langle x\rangle \in R\}\},$$

where $s\langle x\rangle = \langle s_0, \ldots, s_n, x\rangle$.

Lemma 2.4.9. *For all $\mathbf{A}$, $\forall \overrightarrow{v} \in \mathbf{A}$, and for all formula Φ, we have*

$$\{x \in \mathbf{A} \mid \Phi^{\mathbf{A}}(\overrightarrow{v}, x)\} \in \mathcal{D}(\mathbf{A}).$$

Lemma 2.4.10. *For all* $\mathbf{A}$*,*

1. *If* $\mathbf{A}$ *is atransitive, then* $\mathbf{A} \subseteq \mathcal{D}(\mathbf{A})$*,*
2. $\forall X \subseteq \mathbf{A},\ |X| < \omega \to X \in \mathcal{D}(\mathbf{A})$*,*
3. *(with AC)* $|\mathbf{A}| \geq \omega \to |\mathcal{D}(\mathbf{A})| = |\mathbf{A}|$*.*

Proof. Immediate. □

Definition 2.4.11. By transfinite recursion on $\alpha \in \mathbf{ON}$, we define $\mathbf{L}_\alpha$ by

- $\mathbf{L}_0 = \emptyset$,
- $\mathbf{L}_{\alpha+1} = \mathcal{D}(\mathbf{L}_\alpha)$,
- $\mathbf{L}_\alpha = \bigcup\{\mathbf{L}_\gamma \mid \gamma < \alpha\}$, if α is limit.

De denote

$$\mathbf{L} = \bigcup_{\alpha \in \mathbf{ON}} \mathbf{L}_\alpha.$$

Lemma 2.4.12. *Let* $\alpha \in \mathbf{ON}$*. We have*

1. $\mathbf{L}_\alpha$ *is transitive,*
2. $\mathbf{L}_\gamma \subseteq \mathbf{L}_\alpha$*, for all* $\gamma \leq \alpha$*.*

Proof.
1. By induction on α.
2. Immediate.

□

Definition 2.4.13 ($\mathbf{L}$-rank)**.** Let $x \in \mathbf{L}$, and define $\rho(x)$ its $\mathbf{L}$-*rank* to be the least $\alpha \in \mathbf{ON}$ such that $x \in \mathbf{L}$.

Lemma 2.4.14. *We have* $\mathbf{L}_\alpha = \{x \in \mathbf{L} \mid \rho(x) < \alpha\}$*.*

Lemma 2.4.15.

1. $\forall \alpha \in \mathbf{ON}\ \ \mathbf{L}_\alpha \cap \mathbf{ON} = \alpha$*.*
2. $\forall \alpha \in \mathbf{ON}\ \ \alpha \in \mathbf{L} \wedge \rho(\alpha) = \alpha$*.*

Proof.

1. By induction on α.
   - If $\alpha = 0$, then it is trivial, as $L_{(0)} = \emptyset$.

- If $\alpha = \beta + 1$, then by induction hypothesis, $\mathbf{L}_\beta \cap \mathbf{ON} = \beta$. Then, $\mathbf{L}_\beta \subseteq \mathbf{L}_\alpha \subseteq \mathcal{P}(\mathbf{L}_\beta)$, and so $\beta \subseteq \mathbf{L}_\alpha$. Moreover,

$$\begin{aligned}\mathbf{L}_\alpha \cap \mathbf{ON} &\subseteq \mathcal{P}(\mathbf{L}_\beta) \cap \mathbf{ON}\\ &= \alpha\end{aligned}$$

Hence, $\beta \subseteq \mathbf{L}_\alpha \subset \alpha$. We show that $\beta \in L_{(\alpha)}$ to bet that $\alpha = \beta \cup \{\beta\} \subseteq \mathbf{L}_\alpha \cap \mathbf{ON} \subseteq \alpha$. There is a Δ_0^0-formula $\Phi_{\mathbf{ON}}$ that decides the ordinals, and hence absolute for transitive classes.

$$\begin{aligned}\beta &= \{x \in \mathbf{L}_\beta \mid \Phi_{\mathbf{ON}}(x)\}\\ &= \{x \in \mathbf{L}_\beta \mid \Phi_{\mathbf{ON}}^{\mathbf{L}_\beta}(x)\},\end{aligned}$$

hence $\beta \in \mathcal{D}(\mathbf{L}_\beta)$.

- If α is limit, then

$$\begin{aligned}L_{(\alpha)} \cap \mathbf{ON} &= \mathbf{ON} \cap \bigcup_{\beta<\alpha} \mathbf{L}_\beta\\ &= \bigcup(\{\mathbf{L}_\beta \mid \beta < \alpha\} \cap \mathbf{ON})\\ &= \bigcup\{\beta \mid \beta < \alpha\}\\ &= \alpha.\end{aligned}$$

2. We have that $\alpha \cup \{\alpha\} = \alpha + 1 \subseteq \mathbf{L}_{\alpha+1}$, hence $\alpha \in \mathbf{L}_\alpha$. By previous point, $\mathbf{L}_\alpha \cap \mathbf{ON} = \alpha$, and so $\alpha \notin \mathbf{L}_\alpha$. Otherwise, $\alpha \cup \{\alpha\} = \alpha + 1 \subseteq \mathbf{L}_\alpha \cap \mathbf{ON}$, a contradiction.

□

Lemma 2.4.16. *We have* $\mathbf{L}_\alpha \in \mathbf{L}_{\alpha+1}$.

Proof. We have $\mathbf{L}_\alpha = \{x \in \mathbf{L}_\alpha \mid (x = x)^{\mathbf{L}_\alpha}\} \in \mathcal{D}(\mathbf{L}_\alpha)$. □

Lemma 2.4.17. *We have* $\mathbf{L}_\alpha \subseteq \mathbf{V}_\alpha$.

Proof. By induction on α. □

Lemma 2.4.18.

1. $\forall X \subseteq \mathbf{L}_\alpha$, *if* $|X| < \omega$, *then* $X \in \mathbf{L}_{\alpha+1}$.
2. $\forall n \in \omega \;\; \mathbf{L}_n = \mathbf{V}_n$.
3. $\mathbf{L}_\omega = \mathbf{V}_\omega$.

Proof.

1. Obvious.

2. By point 1..
3. Remark that $\mathbf{L}_\omega = \bigcup_{n<\omega} \mathbf{L}_n = \bigcup_{n<\omega} \mathbf{V}_n = \mathbf{V}_\omega$.

□

Lemma 2.4.19. *With the axiom of choice.* $\forall \alpha \geq \omega \ \ |\mathbf{L}_\alpha| = |\alpha|$.

Proof. We know that $\alpha \subseteq \mathbf{L}_\alpha$, hence $|\alpha| \leq |\mathbf{L}_\alpha|$. Conversely, we show that $|\mathbf{L}_\alpha| \leq |\alpha|$. by induction on $\alpha \geq \omega$.

- If $\alpha = \omega$, it is obvious, as $\mathbf{L}_\omega = \omega$.
- If $\alpha = \beta + 1$, then

$$\begin{aligned} |\mathbf{L}_\alpha| &= |\mathcal{D}(\mathbf{L}_\beta)| \\ &= |\mathbf{L}_\beta^{<\omega}| \\ &= |\mathbf{L}_\beta| \\ &= |\beta| && \text{I.H.} \\ &= |\beta + 1| \\ &= |\alpha|. \end{aligned}$$

- If α is limit, then

$$\begin{aligned} |\alpha| &\leq |\mathbf{L}_\alpha| \\ &= \left| \bigcup_{\beta<\alpha} \mathbf{L}_\beta \right| \\ &\leq |\alpha| \otimes |\alpha| \\ &= |\alpha|. \end{aligned}$$

□

Theorem 2.4.20 (Reflextion theorem)**.** *For each sequence of fomulas* $\Phi_1, \ldots, \Phi_n$*, and for all* $\alpha \in$ **ON***, there exists* $\beta > \alpha$ *such that* $\Phi_1, \ldots, \Phi_n$ *are absolute for* $\mathbf{L}_\beta$ *and* $\mathbf{L}$.

Proof. We may assume that the sequance $\Phi_1, \ldots, \Phi_n$ is closed under subformulas. For each $i = 1, \ldots, n$, we define $F_i : \mathbf{ON} \longrightarrow \mathbf{ON}$.

- If Φ_i is of the form $\exists x \ \ \Phi_j(x, y_1, \ldots, y_l)$, define

$$G_i(y_1, \ldots, y_l) = \begin{cases} 0 & \text{if } (\neg \exists x \ \ \Phi_j(x, \overrightarrow{y}))^{\mathbf{L}} \\ \eta & \text{otherwise,} \end{cases}$$

  where η is the least ordinal such that $(\exists x \in \mathbf{L}_\eta \ \ \Phi_j(x, \overrightarrow{y}))^{\mathbf{L}}$. If $(\exists x \ \ \Phi_j(x, \overrightarrow{y}))^{\mathbf{L}}$, set

$$F_i(\beta) = \sup_{\overrightarrow{y} \ni \mathbf{L}_\beta} G_i(\overrightarrow{y}).$$

- If Φ_i is not of the previous form, define $F_i(\beta) = 0$.

We fix α, and define a sequence $\beta_0, \beta_1, \dots$:

- $\beta_0 = \alpha$,
- $\beta_{p+1} = \max\{\beta_p + 1, F_1(\beta_p), F_2(\beta_p), \dots, F_n(\beta_p)\}$.

Define $\beta = \sup_{p\in\omega} \beta_p$. We have that $\alpha < \beta$, and that β is a limit ordinal. If $\gamma < \beta$, then $\gamma < \beta_p$, for some $p \in \omega$. Hence,

$$F_i(\gamma) \leq F_i(\beta_p) \leq \beta_{p+1} < \beta$$

and so F_i is increasing. We need to check that for each $0 \leq i \leq n$, Φ_i is of the form

$$\forall \overrightarrow{y} \in \mathbf{L}_\beta \ \ (\Phi_i^{\mathbf{L}_\beta}(\overrightarrow{y}) \leftrightarrow \Phi_i^{\mathbf{L}}(\overrightarrow{y}).$$

We prove this by induction on the height of Φ_i. The only case we have to check is if $\Phi_i = \exists x \ \ \Phi_j(\overrightarrow{y})$, then

$$\forall \overrightarrow{y} \in \mathbf{L}_\beta \ \ (\exists x \ \ \Phi_j(x, \overrightarrow{y}))^{\mathbf{L}} \to (\exists x \ \ \Phi_j(x; \overrightarrow{y}))^{\mathbf{L}_\beta}.$$

We fix $\overrightarrow{y} = y_1, \dots, y_l \in \mathbf{L}_\beta$. So for some $p \in \omega$, we have $y_1, \dots, y_l \in \mathbf{L}_{\beta_p}$. By construction, there exists $x \in \mathbf{L}_{\beta_{p+1}}$ such that $\Phi_j(x, \overrightarrow{y})$. By induction hypothesis, $\Phi_j(x, \overrightarrow{y})^{\mathbf{L}_\beta}$, hence $(\exists x \ \ \Phi_j(x, \overrightarrow{y}))^{\mathbf{L}_\beta}$. □

Theorem 2.4.21. *In ZFC.* **L** *is a model of ZF.*

Proof.

- The axiom of extensionality holds because **L** is transitive.
- The axiom of fundation holds because $\mathbf{L} \subseteq \mathbf{WF}$.
- We show that the comprehension axiom holds by showing that for all formula Ψ, $\forall z \in \mathbf{L}$, $\forall \overrightarrow{v} \in \mathbf{L}$, we have $\{x \in z \mid \Psi^{\mathbf{L}}(x, z, \overrightarrow{v})\} \in \mathbf{L}$. Let $\alpha \in \mathbf{ON}$ be such that $z, \overrightarrow{v} \in \mathbf{L}_\alpha$. We need to show that there exists $\beta > \alpha$ such that Ψ is absolute for $\mathbf{L}_\beta$, i.e.

  $$(x \in z \wedge \Psi^{\mathbf{L}}(x, z, \overrightarrow{v}) \leftrightarrow (x \in z \wedge \Psi^{\mathbf{L}_\beta}(x, z, \overrightarrow{v}).$$

  Finally, $\{x \in z \mid \Psi_{\mathbf{L}}(x, z, \overrightarrow{v})\} = \{x \in \mathbf{L}_\beta \mid \Phi^{\mathbf{L}_\beta}(x, z, \overrightarrow{v})\}$, where $\Phi = (x \in z \wedge \Psi)$. We make use of the reflexion theorem by considering the sequence $\Phi_1, \dots, \Phi_l$ of subformulas of Φ. We then take β as defined by the proof of the theorem.

- The axiom of pairing and union are easy to prove.
- We prove the powerset axiom. Remark that

  $$\forall x \in \mathbf{L} \exists y \in \mathbf{L} \forall z \in L \ \ (\underbrace{(z \subseteq x)^{\mathbf{L}}}_{=(z \cap \mathbf{L} \subseteq x)} \to z \in y).$$

  Let $\alpha = \sup\{\rho(z) + 1 \mid z \in \mathbf{L} \wedge z \subseteq x\}$. If $z \in \mathbf{L} \wedge z \subseteq x$, then $z \in \mathbf{L}_\alpha$.

- We prove the axiom of replacement. Assume that $A \in \mathbf{L}$, $\overrightarrow{w} \in \mathbf{L}$, and that $\forall x \in A \exists! y \ \ \Phi^{\mathbf{L}}(x, y, A, \overrightarrow{w})$. Let $\alpha = \sup\{\rho(y)+1 \mid \exists x \in A \ \ \Phi^{\mathbf{L}}(x, y, A, \overrightarrow{w})\}$. Then $\mathbf{L}_\alpha$ satisfies $\{y \mid \exists x \in A \ \ \Phi^{\mathbf{L}}(x, y, A, \overrightarrow{w})\} \subseteq \mathbf{L}_\alpha \in \mathbf{L}$.

- The axiom of infinity is trivial, as $\omega \in \mathbf{L}$.

□

Theorem 2.4.22. $\mathbf{L}$ *is an* inner model *of ZFC, i.e.*

1. $\mathbf{L}$ *is transitive,*

2. $\mathbf{ON} \subseteq \mathbf{L}$,

3. $(ZF)^{\mathbf{L}}$.

Remark 2.4.23. $\mathbf{L}$ is the smallest inner model of ZF, because for every other inner model $\mathbf{M}$, we have $\mathbf{L}_\alpha^{\mathbf{M}} = \mathbf{L}_\alpha$, hence $\mathbf{L}^{\mathbf{M}} = \mathbf{L}$.

Definition 2.4.24. Define $\lhd_\alpha$ by transfinite recursion :

- $\lhd_0 = \emptyset$.

- Assume that $\lhd_\alpha$ well orders $\mathbf{L}_\alpha$, and denote by $\lhd_\alpha^n$ the lexicographical order induces by $\lhd_\alpha$. If $x \in \mathbf{L}_{\alpha+1} = \mathcal{D}(\mathbf{L}_\alpha)$, let n_x be the smallest $n \in \omega$ such that

$$\exists s \in \mathbf{L}_\alpha^n \exists R \in D_f(\mathbf{L}_\alpha, n+1) \ \ x = \{y \in \mathbf{L}_\alpha \mid s\langle y \rangle \in R\}.$$

  Let s_x be the least element in $\mathbf{L}_\alpha^{n_x}$ (with respect to $\lhd_\alpha^{n_x}$) such that

$$\exists R \in D_f(\mathbf{L}_\alpha, n+1) \ \ x = \{y \in \mathbf{L}_\alpha \mid s_x\langle y \rangle \in R\}.$$

  Take m_x to be the least integer $m \in \omega$ such that

$$x = \{y \in \mathbf{L}_\alpha \mid s_x\langle y \rangle \in En(m, \mathbf{L}_\alpha, n_x)\}.$$

  For $x, y \in \mathbf{L}_{\alpha+1}$, define $\lhd_{\alpha+1}$ by

$$x \lhd_{\alpha+1} y \iff \begin{cases} x, y \in \mathbf{L}_\alpha \text{ and } x \lhd_\alpha y, \text{ or} \\ x \in \mathbf{L}_\alpha \text{ and } y \notin \mathbf{L}_\alpha, \text{ or} \\ x, y \notin \mathbf{L}_\alpha \text{ and } (n_x, s_x, m_x) \prec (n_y, s_y, m_y), \end{cases}$$

  where $\prec$ denotes the lexicographical order.

- If α is limit, then

$$\lhd_\alpha = \{\langle x, y \rangle \in \mathbf{L}_\alpha \times \mathbf{L}_\alpha \mid (\rho(x), x) \prec (\rho(y), y)\},$$

  where $\prec$ is here the lexicographical order, i.e. $\rho(x) < \rho(y)$, or $\rho(x) = \rho(y)$ and $x \lhd_{\rho(x)+1} y$.

Proposition 2.4.25. *For all* $\alpha \in \mathbf{ON}$, $\lhd_\alpha$ *is a well ordering on* $\mathbf{L}_\alpha$.

Proof. Immediate. □

Definition 2.4.26. Define

$$x <_{\mathbf{L}} y \iff \begin{cases} \rho(x) < \rho(y), \text{ or} \\ \rho(x) = \rho(y) \text{ and } x \lhd_{\rho(x)+1} y. \end{cases}$$

Axiom 2.4.27 (Axiom of constructibility)**.** The axiom of constructibility, or "$\mathbf{V} = \mathbf{L}$" is the following statement :

$$\forall x \exists \alpha \in \mathbf{ON} \;\; x \in \mathbf{L}_\alpha.$$

Theorem 2.4.28. *ZF* $\vdash \mathbf{V} = \mathbf{L} \to AC$.

Proof. If $\mathbf{V} = \mathbf{L}$, then $\forall x \in \mathbf{V}$, $\exists \alpha \in \mathbf{ON}$ such that $s \in \mathbf{L}_\alpha$, and so x is well ordered by $\lhd_\alpha$. □

Definition 2.4.29 (Elementary submodel)**.** For $\mathbf{X}, \mathbf{M} \in \mathbf{V}$, we say that $\mathbf{X} \prec \mathbf{M}$ ($\mathbf{X}$ is an *elementary submodel* of $\mathbf{M}$, or $\mathbf{M}$ is an *elementary extension* of $\mathbf{X}$) if

1. $\mathbf{X} \subseteq \mathbf{M}$,
2. $\forall \Phi$ a formula, $\forall \overrightarrow{x} \in \mathbf{X}$, we have
$$\Phi^{\mathbf{X}}(\overrightarrow{x}) \leftrightarrow \Phi^{\mathbf{M}}(\overrightarrow{x}).$$

Lemma 2.4.30. *Let $\alpha > \omega$ be a limit ordinal. If $\mathbf{X} \prec \mathbf{L}_\alpha$, then there exists an isomorphism $(\mathbf{X}, \in) \xrightarrow{\cong} (\mathbf{L}_\beta, \in)$, for some $\beta \leq \alpha$.*

Proof. We consider the Mostovski collapse of $\mathbf{X}$: $\pi : (\mathbf{X}, \in) \xrightarrow{\cong} (\mathbf{M}, \in)$. We have that $\mathbf{X}$ is extensional, as $\mathbf{L}_\alpha$ is. Therefore, $\mathbf{M}$ is extensional and transitive. We look at $\pi^{-1} : \mathbf{M} \longrightarrow \mathbf{X} \hookrightarrow \mathbf{L}_\alpha$. For $\overrightarrow{x} \in \mathbf{M}$, we have

$$\Phi^{\mathbf{M}}(\overrightarrow{x}) \leftrightarrow \Phi^{\mathbf{X}}(\pi^{-1}(\overrightarrow{x})) \leftrightarrow \Phi^{\mathbf{L}_\alpha}(\pi^{-1}(\overrightarrow{x})).$$

Hence, π^{-1} is an elementary injection, and $\pi^{-1}[\mathbf{M}] \prec \mathbf{L}_\alpha$. For all $\gamma \leq \alpha$, the following formula holds :

$$\exists v \in \mathbf{L}_\alpha \;\; v = \mathbf{L}_\gamma,$$

and the later is Δ_0^0, and so of the form $(\forall \gamma \in \mathbf{ON} \exists v \exists x \;\; \Psi(x, \gamma, v))^{\mathbf{L}_\alpha}$. Hence $(\forall \gamma \in \mathbf{ON} \exists v \exists x \;\; \Psi(x, \gamma, v))^{\mathbf{M}}$, i.e.

$$\forall \gamma \in \mathbf{ON} \cap \mathbf{M} \exists v \in \mathbf{M} \exists x \in \mathbf{M} \;\; \Psi(x, \gamma, v).$$

Moreover, $\beta = \mathbf{ON} \cap \mathbf{M}$ is an ordinaln by transitivity of $\mathbf{M}$. Notice that β is a limit ordinal because

$$\begin{aligned} \alpha \text{ limit} &\implies \forall \gamma \in \alpha \exists \gamma' \in \alpha \;\; \gamma \in \gamma' \\ &\implies (\forall \gamma \exists \gamma' \;\; \gamma \in \gamma')^{\mathbf{L}_\alpha} \\ &\implies (\forall \gamma \exists \gamma' \;\; \gamma \in \gamma')^{\mathbf{M}}. \end{aligned}$$

We have that $\forall \gamma \in \beta$, $\mathbf{L}_\gamma \in \mathbf{M}$, hence

$$\mathbf{L}_\beta = \bigcup_{\gamma < \beta} \mathbf{L}_\gamma \subseteq \mathbf{M}.$$

Conversely, we have that $\mathbf{L}_\alpha = \bigcup_{\gamma<\alpha} \mathbf{L}_\gamma$, i.e.

$$\forall x \in \mathbf{L}_\alpha \exists u \in \mathbf{L}_\alpha \exists \gamma \in \alpha \quad \underbrace{u = \mathbf{L}_\alpha}_{\exists z \in \mathbf{L}_\alpha \ \underbrace{\Psi(z,\gamma,u)}_{\in \Delta_0^0}} \quad \wedge x \in u,$$

i.e. $(\forall x \exists u \exists \gamma \exists z \ \ \Psi(z,\gamma,u) \wedge x = u)^{\mathbf{L}_\alpha}$. Hence $(\forall x \exists u \exists \gamma \exists z \ \ \Psi(z,\gamma,u) \wedge x = u)^{\mathbf{M}}$, i.e. $\forall x \in \mathbf{M} \exists u \in \mathbf{M} \exists \gamma \in \mathbf{M}(\exists z \in \mathbf{M} \ \ \Psi^{\mathbf{M}}(z,\gamma,u) \wedge x = u)$, i.e.

$$\forall x \in \mathbf{M} \exists u \in \mathbf{M} \exists \gamma \in \mathbf{M}(u = \mathbf{L}_\gamma \wedge x = u)^{\mathbf{M}}.$$

Hence, $\mathbf{M} \subseteq \mathbf{L}_\beta$, and so we have equality. ☐

Theorem 2.4.31 (Tarski citerion)**.** *Let $\mathbf{X}, \mathbf{M} \in \mathbf{V}$. We have that $\mathbf{X} \prec \mathbf{M}$ if and only if for all formula Φ*

$$\forall \overrightarrow{y} \in \mathbf{M} \ \ (\exists x \ \ \Phi(x, \overrightarrow{y}))^{\mathbf{M}} \to (\exists x \ \ \Phi(x, \overrightarrow{y}))^{\mathbf{X}}.$$

Proof. By induction on the height of Φ. ☐

Lemma 2.4.32. *For $A \subseteq \mathbf{L}_\alpha$, there exists M such that $A \subseteq M$, $M \prec \mathbf{L}_\alpha$, and $|M| = \max\{|A|, \aleph_0\}$.*

Proof. Define

- $M_0 = A \cup \omega$,
- $$M_{n+1} = M_n \cup \{x \in \mathbf{L}_\alpha \mid \exists \overrightarrow{v} \in M_n \text{ st there exists } \Phi \text{ st } \Phi(x, \overrightarrow{v})^{\mathbf{L}_\alpha} \wedge \forall y \in \mathbf{L}_\alpha \ \ (\Phi(y, \overrightarrow{v}) \to y \not<_{\mathbf{L}} x)\},$$
- $M = \bigcup_{n\in\omega} M_n$.

We have that M satisfies the Tarski criterion, hence $M \prec \mathbf{L}_\alpha$. Moreover,

$$|M_{n+1}| = \max\{|M_n|, |M_n^{<\omega}|\} = |M_n|,$$

as M_n is already infinite. Hence,

$$|M| = |M_0| = \max\{|A|, \aleph_0\}.$$

☐

Lemma 2.4.33. *If $\mathbf{V} = \mathbf{L}$, then for all infinite cardinal λ, we have $\mathcal{P}(\lambda) \subseteq \mathbf{L}_{\lambda^+}$.*

Proof. Let $X \in \mathcal{P}(\lambda)$. We have that $X \in \mathbf{L}_\alpha$, for some $\alpha \in \mathbf{ON}$. Define $A = \lambda \cup \{X\}$. There exists M such that $A \subseteq M \prec \mathbf{L}_\alpha$, and $|M| = \max\{|A|, \aleph_0\} = \lambda$. We have the Mostovski collapse $\pi : (M, \in) \xrightarrow{\cong} (\mathbf{L}_\beta, \in)$, for some $\beta \leq \alpha$. Since λ is transitive, $\pi|_\lambda$ is the identity. Hence

$$\begin{aligned} \pi[X] &= \{\pi(\gamma) \mid \gamma \in X\} \\ &= \{\gamma \mid \gamma \in X\} \\ &= X. \end{aligned}$$

So $|M| = \lambda$, hence $|M| = |\mathbf{L}_\beta| = \lambda$. So $|\beta| \leq |\mathbf{L}_\beta| \leq \lambda$, hence $\beta < \lambda^+$, and $X \in \mathbf{L}_{\lambda^+}$. We have shown that $\mathcal{P}(\lambda) \subseteq \mathbf{L}_{\lambda^+}$. ☐

Theorem 2.4.34. *If* $\mathbf{V} = \mathbf{L}$*, then GCH holds.*

Proof. If $\mathbf{V} = \mathbf{L}$, and λ is an infinite cardinal, than $\mathcal{P}(\lambda) \subseteq \mathbf{L}_{\lambda^+}$, therefore

$$\begin{aligned} \lambda &< |\mathcal{P}(\lambda)| \\ &\leq |\mathbf{L}_{\lambda^+}| \\ &= \lambda^+. \end{aligned}$$

So $|\mathcal{P}(\lambda)| = 2^\lambda = \lambda^+$. □

Theorem 2.4.35. *In ZFC. Let* Φ *be a closed formula. For all transitive set* X*, there exists* Y *such that*

$$X \subseteq Y \wedge |Y| \leq \max\{|X|, \aleph_0\} \wedge \Phi \leftrightarrow \Phi^Y.$$

Proof. Exercise. □

Corollary 2.4.36. *ZFC is not finitely axiomatizable.*

Proof. Otherwise, we could build a set model of ZFC, which is impossible by Gödel's second incompleteness theorem. □

Chapter 3

Forcing

See prof. Duparcs paper about forcing.

The ZFC axioms

Axiom (0. Set existence)**.**

$$\exists x \ \ x = x.$$

Axiom (1. Extensionality)**.**

$$\forall x \forall y \ \ (\forall z \ \ (z \in x \leftrightarrow z \in y) \to x = y).$$

Axiom (2. Comprehension schema)**.**

$$\forall z \forall \vec{w} \exists y \forall x \ \ (x \in y \leftrightarrow x \in z \wedge \Phi,$$

where $\Phi = \Phi(x, z, \vec{w})$ is any formula with free variables among $x, z, w_1, \ldots, w_n$.

Axiom (3. Pairing)**.**

$$\forall x \forall y \exists z \ \ x \in z \wedge y \in z.$$

Axiom (4. Union)**.**

$$\forall a \exists b \forall x \forall y \ \ x \in y \wedge y \in a \to x \in b.$$

Axiom (5. Infinity)**.**

$$\exists x \ \ (\emptyset \in x \wedge \forall y \ \ (y \in x \to y \cup \{y\} \in x)).$$

Axiom (6. Power set)**.**

$$\forall x \exists y \forall z \ \ (\forall u \ \ (u \in z \to u \in x) \to z \in y).$$

Axiom (7. Replacement schema)**.**

$$\forall A \forall \vec{w} \ \ [\forall x \ \ (x \in A \to \exists! y \ \ \Phi) \to \exists Y \forall x \ \ (x \in A \to \exists y \ \ (y \in Y \wedge \Phi))],$$

where $\Phi = \Phi(x, y, A, \vec{w})$ is any formula with free variables among $x, y, A, w_1, \ldots, w_n$, and where $\exists! y \ \ \Phi$ abbreviates

$$\exists y \ \ \Phi(x, y, A, \vec{w}) \wedge (\forall z \ \ \Phi(x, z, A, \vec{w}) \to z = y).$$

Axiom (8. Foundation)**.**

$$\forall x \ \ (\exists y \ \ y \in x) \to (\exists y \ \ y \in x \wedge \neg(\exists z \ \ z \in x \wedge z \in y)).$$

Axiom (9. Choice)**.**

$$\forall x \exists c \forall z \exists y \forall u \ \ z \in x \to [y \in z \wedge y \in c \wedge ((u \in z \wedge u \in c) \to u = y)].$$

Index

Bibliography

[1] Kenneth Kunen, *Set theory, an introduction to independence proof*, 1980.

[2] Thomas Jech, *Set theory.*

[3] Jean-Louis Krinir, *Théorie des ensembles.*