ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Quasi categories

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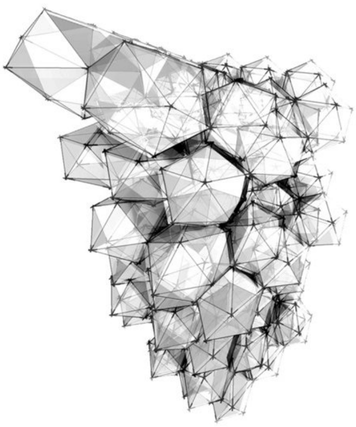

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Chapter 1

Categorical prerequisites

In this whole project, the term "category" will be used to denote small categories, that is, categories with a set of objects (rather that a class), and a set of morphisms between any two objects.

1.1 Generators and relations

Free categories. Let $G = (V, E, s, t)$ be an oriented multigraph, with V the set of its vertices, E the set of its edges, $s, t : E \longrightarrow V$ the source and target maps respectively. The *free category* $\mathscr{F}G$ generated by G is defined as follow :

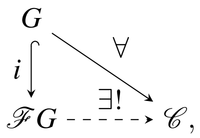
$$\begin{aligned} \operatorname{ob} \mathscr{F}G &= V, \\ \hom \mathscr{F}G &= \{w_1 \cdots w_n \in E^{<\omega} \mid t(w_i) = s(w_{i+1}), \forall 1 \leq i < n\}, \end{aligned}$$

where compositions of morphisms are given by concatenation of words, and $\forall A \in \operatorname{ob} \mathscr{F}$, the identity morphism 1 is the empty word.

If $m : G \longrightarrow G'$ is a graph homomorphism, then we can define a functor

$$\begin{aligned} \mathscr{F}m : \mathscr{F}G &\longrightarrow \mathscr{F}G' \\ v &\longmapsto m(v) && \forall v \operatorname{ob} \mathscr{F}G, \\ w_1 \cdots w_n &\longmapsto m(w_1) \cdots m(w_n) && \forall w_1 \cdots w_n \in \hom \mathscr{F}G, \end{aligned}$$

which can easily be proven well defined. Thereby, $\mathscr{F} : \mathscr{G}raph \longrightarrow \mathscr{C}at$ is a functor, and as expected from the denomination "free", it is left adjoint to the forgetful functor $U : \mathscr{C}at \longrightarrow \mathscr{G}raph$. In particular, we have a unique lifting property that reads as follow :



where i is the obvious inclusion of multigraphs, and $\mathscr{C}$ any category.

Quotient categories. Let $\mathscr{C}$ be a category. A *congruence relation* $\simeq$ on $\mathscr{C}$ consists in an equivalence relation $\simeq_{A,B}$ on $[A,B]_{\mathscr{C}}$, $\forall A,B \in \mathrm{ob}\mathscr{C}$, that is compatible with composition, i.e. $\forall f, f' : A \longrightarrow B$, $\forall g, g' : B \longrightarrow C$, we have

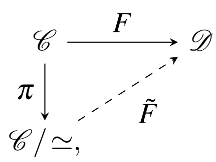
$$\begin{cases} f \simeq_{A,B} f' \implies gf \simeq_{A,C} gf', \\ g \simeq_{B,C} g' \implies gf \simeq_{A,C} g'f. \end{cases}$$

The *quotient category* $\mathscr{C}/\simeq$ is defined by

$$\begin{aligned} &\mathrm{ob}(\mathscr{C}/\simeq) = \mathrm{ob}\mathscr{C}, \\ &[A,B]_{\mathscr{C}/\simeq} = [A,B]_{\mathscr{C}}/\simeq_{A,B} \qquad\qquad \forall A,B \in \mathrm{ob}\mathscr{C}. \end{aligned}$$

Quotient categories enjoy the following universal propery :

Theorem 1.1.1. *Let $\mathscr{C}$ be a category, $\simeq$ a congruence relation on $\mathscr{C}$, and $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor such that $Ff = Fg$, for all parallel arrows $f \simeq g$ in $\mathscr{C}$. Then $\exists! \tilde{F} : \mathscr{C}/\simeq \longrightarrow \mathscr{D}$ such that the following diagram commutes :*



where π is the obvious projection functor.

Every category $\mathscr{C}$ is the quotient of a free category by an appropriate congruence relation, which is called a definition by generator and relations of $\mathscr{C}$. Indeed, $\mathscr{C} = (\mathrm{ob}\mathscr{C}, \hom\mathscr{C}, \mathrm{dom}, \mathrm{codom})$ can be seen as an oriented graph, from wich we build the free category $\mathscr{F}\mathscr{C}$. We define now a congruence relation $\simeq$ on $\mathscr{F}\mathscr{C}$ as follow : $\forall A,B \in \mathrm{ob}\mathscr{F}\mathscr{C}$, $\forall v_1 \cdots v_m, w_1 \cdots w_n \in [A,B]_{\mathscr{F}\mathscr{C}}$,

$$v_1 \cdots v_m \simeq_{A,B} w_1 \cdots w_n \iff v_1 \circ \cdots \circ v_m = w_1 \circ \cdots \circ w_n.$$

Using the previous theorem, it is then clear that $\mathscr{C} = \mathscr{F}\mathscr{C}/\simeq$.

1.2 Enriched categories

Monoidal categories. A *monoidal category* $\mathscr{M}$ is a classical category endowed with a functor $\otimes : \mathscr{M} \times \mathscr{M} \longrightarrow \mathscr{M}$, the *tensor product*, such that :

1. $\otimes$ is associative up to natural isomorphims, i.e. there exists a natural isomorphism $\alpha : (-_1 \otimes -_2) \otimes -_3 \xrightarrow{\cong} -_1 \otimes (-_2 \otimes -_3)$ such that $\forall A, B, C, D \in \mathrm{ob}\,\mathscr{M}$, the following diagram commutes :

$$\begin{array}{ccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha} & (A \otimes B) \otimes (C \otimes D) \\ \downarrow{\scriptstyle \alpha \otimes 1} & & \\ (A \otimes (B \otimes C)) \otimes D & & \downarrow{\scriptstyle \alpha} \\ \downarrow{\scriptstyle \alpha} & & \\ A \otimes ((B \otimes C) \otimes D) & \xrightarrow{1 \otimes \alpha} & A \otimes (B \otimes (C \otimes D)), \end{array}$$

2. $\otimes$ admits an objet $I \in \mathrm{ob}\,\mathscr{M}$ as a left and right neutral element up to some natural isomorphisms, i.e. there exists natural isomorphisms $\rho : - \otimes I \xrightarrow{\cong} 1$, $\lambda : I \otimes - \xrightarrow{\cong} 1$, such that $\forall A, B \in \mathrm{ob}\,\mathscr{M}$, the following diagram commutes :

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha} & A \otimes (I \otimes B) \\ & {\scriptstyle \rho \otimes B} \searrow \quad \swarrow {\scriptstyle A \otimes \lambda} & \\ & A \otimes B. & \end{array}$$

The category $\mathscr{M}$ is said *symmetric* if there is a natural isomorphism $\tau : -_1 \otimes -_2 \xrightarrow{\cong} -_2 \otimes -_1$. It is *strictly monoidal* if the natural isomorphisms α, ρ and λ are equalities. It is *strictly symmectric monoidal* if it is symmetric, and τ is the equality.

The actual definition. An *enriched category* $\mathscr{E}$ over $\mathscr{M}$, or $\mathscr{M}$-category, consists in

1. a set $\mathrm{ob}\,\mathscr{E}$ of objects,

2. $\forall A, B \in \mathrm{ob}\,\mathscr{E}$, an object $[A, B]_{\mathscr{E}} \in \mathrm{ob}\,\mathscr{M}$,

3. $\forall A, B, C \in \mathrm{ob}\,\mathscr{E}$, a morphism $\circ = \circ_{A,B,C} : [B, C]_{\mathscr{E}} \otimes [A, B]_{\mathscr{E}} \longrightarrow [A, C]_{\mathscr{E}}$ that is

associative up to α, i.e. $\forall D \in \mathrm{ob}\,\mathscr{E}$, the following diagram commutes :

$$\begin{array}{ccc}
([C,D]_{\mathscr{E}} \otimes [B,C]_{\mathscr{E}}) \otimes [A,B]_{\mathscr{E}} & \xrightarrow{\circ \otimes 1} & [B,D]_{\mathscr{E}} \otimes [A,B]_{\mathscr{E}} \\
 & & \downarrow \circ \\
\alpha \Big\downarrow & & [A,D]_{\mathscr{E}} \\
 & & \uparrow \circ \\
[C,D]_{\mathscr{E}} \otimes ([B,C]_{\mathscr{E}} \otimes [A,B]_{\mathscr{E}}) & \xrightarrow{1 \otimes \circ} & [C,D]_{\mathscr{E}} \otimes [A,C]_{\mathscr{E}},
\end{array}$$

4. $\forall A \in \mathrm{ob}\,\mathscr{E}$, a morphism $1 : I \longrightarrow [A,A]_{\mathscr{E}}$ such that $\forall B \in \mathrm{ob}\,\mathscr{E}$, the following diagrams commute :

$$\begin{array}{ccc}
I \otimes [A,B]_{\mathscr{E}} & \xrightarrow{1 \otimes 1} & [B,B]_{\mathscr{E}} \otimes [A,B]_{\mathscr{E}} \\
 & \searrow^{\lambda} & \downarrow \circ \\
 & & [A,B]_{\mathscr{E}},
\end{array}$$

$$\begin{array}{ccc}
[A,B]_{\mathscr{E}} \otimes I & \xrightarrow{1 \otimes 1} & [A,B]_{\mathscr{E}} \otimes [A,A]_{\mathscr{E}} \\
 & \searrow^{\rho} & \downarrow \circ \\
 & & \mathscr{E}.
\end{array}$$

Let $\mathscr{E}'$ be another $\mathscr{M}$-category. An enriched functor, or $\mathscr{M}$-functor, $F : \mathscr{E} \longrightarrow \mathscr{E}'$, consists in

1. a map $F_0 : \mathrm{ob}\,\mathscr{E} \longrightarrow \mathrm{ob}\,\mathscr{E}'$, and we'll denote by $FA = F_0(A)$, if $A \in \mathrm{ob}\,\mathscr{E}$,
2. $\forall A, B \in \mathrm{ob}\,\mathscr{E}$, a morphism $F = F_{A,B} : [A,B]_{\mathscr{E}} \longrightarrow [FA,FB]_{\mathscr{E}'}$ in $\mathscr{M}$, that is compatible with compositions and identities, i.e. such that $\forall C \in \mathrm{ob}\,\mathscr{E}$, the following diagrams commute :

$$\begin{array}{ccc}
[B,C]_{\mathscr{E}} \otimes [A,B]_{\mathscr{E}} & \xrightarrow{\circ} & [A,C]_{\mathscr{E}} \\
F \otimes F \Big\downarrow & & \Big\downarrow F \\
[FB,FC]_{\mathscr{E}'} \otimes [FA,FB]_{\mathscr{E}'} & \xrightarrow{\circ} & [FA,FC]_{\mathscr{E}'},
\end{array}$$

$$\begin{array}{ccc}
 & I & \\
{}^{1}\swarrow & & \searrow^{1} \\
[A,A]_{\mathscr{E}} & \xrightarrow{F} & [FA,FB]_{\mathscr{E}'}.
\end{array}$$

Enriched approach to n-categories. Define a 0-category to be a set, a 0-functor to be a set map, and denote $\mathscr{C}at_0 = \mathscr{S}et$ the category of all 0-categories. Endowing it with the cartesian product makes it into a strictly symmetric monoidal category, where the neutral element is the singleton $\star_0 = \{*\}$.

Assume now that $\mathscr{C}at_n$ is strictly symmetric monoidal, with tensor product $\times_n$ and neutral element $\star_n$, and define a $(n+1)$-*category* to be a category enriched over $\mathscr{C}at_n$. Define $\mathscr{C}at_{n+1}$ to be the category of all such categories and enriched functors. Consider the obvious product $\times_{n+1} : \mathscr{C}at_{n+1} \times \mathscr{C}at_{n+1} \longmapsto \mathscr{C}at_{n+1}$, and the $(n+1)$-category $\star_{n+1}$ such that $\mathrm{ob}\,\star_{n+1} = \{*\}$, and $[*,*]_{\star_{n+1}} = \star_n$. The latter is clearly a left and right neutral element for $\times_{n+1}$, and thus $\mathscr{C}at_{n+1}$ can be made into a strict symmetric monoidal category.

From there, we define by induction the category $\mathscr{C}at_n$ of n-categories, $\forall n \in \mathbb{N}$. For convenience, we will loose the indices under $\times$ and $\star$.

1.3 Double categories

A *double category* $\mathscr{D}$ consists in a *horizontal category* $H\mathscr{D}$ and a *vertical category* $V\mathscr{D}$ endowed with functors

$$H\mathscr{D} \times_{V\mathscr{D}} H\mathscr{D} \xrightarrow{\ \diamond\ } H\mathscr{D} \overset{d}{\underset{c}{\rightrightarrows}} V\mathscr{D}, \qquad H\mathscr{D} \xleftarrow{\ i\ } V\mathscr{D},$$

where $H\mathscr{D} \times_{V\mathscr{D}} H\mathscr{D}$ is the following pullback :

$$\begin{array}{ccc} H\mathscr{D} \times_{V\mathscr{D}} H\mathscr{D} & \longrightarrow & H\mathscr{D} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle d} \\ H\mathscr{D} & \xrightarrow{\ c\ } & V\mathscr{D}. \end{array}$$

We define

- the objects of $\mathscr{D}$ as being the objects of $V\mathscr{D}$,
- the horizontal morphisms of $\mathscr{D}$ as being the objects of $H\mathscr{D}$,
- the vertical morphisms of $\mathscr{D}$ as being the morphisms of $V\mathscr{D}$,

- the squares of $\mathscr{D}$ as being the morphisms of $H\mathscr{D}$, which we'll represent by

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ j\downarrow & \alpha & \downarrow k \\ \bullet & \xrightarrow[g]{} & \bullet \end{array},$$

  where f and g are horizontal arrows, $\alpha : f \longrightarrow g$, and $j = d(\alpha)$, $k = c(\alpha)$ are vertical arrows.

We define horizontal and vertical (associative) composition of squares as follows :

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ j\downarrow & \alpha & \downarrow k \\ \bullet & \xrightarrow[g]{} & \bullet \end{array} \quad \begin{array}{ccc} \bullet & \xrightarrow{f'} & \bullet \\ k\downarrow & \beta & \downarrow l \\ \bullet & \xrightarrow[g']{} & \bullet \end{array} \quad \longmapsto \quad \begin{array}{ccc} \bullet & \xrightarrow{(f\,f')} & \bullet \\ j\downarrow & (\alpha\,\beta) & \downarrow l \\ \bullet & \xrightarrow[(g\,g')]{} & \bullet \end{array},$$

$$\begin{array}{c} \begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ k\downarrow & \alpha & \downarrow l \\ \bullet & \xrightarrow[g]{} & \bullet \end{array} \\ \begin{array}{ccc} \bullet & \xrightarrow{g} & \bullet \\ k'\downarrow & \gamma & \downarrow l' \\ \bullet & \xrightarrow[h]{} & \bullet \end{array} \end{array} \quad \longmapsto \quad \begin{array}{ccc} \bullet & \xrightarrow{g} & \bullet \\ \begin{pmatrix} k \\ k' \end{pmatrix}\downarrow & \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} & \downarrow\begin{pmatrix} l \\ l' \end{pmatrix} \\ \bullet & \xrightarrow[h]{} & \bullet \end{array},$$

where $(x\,y) = y \diamond x$, and $\begin{pmatrix} x \\ y \end{pmatrix} = yx$ is the composition of morphisms in $H\mathscr{D}$ or $V\mathscr{D}$, depending on the context. Those laws admit the following horizontal and vertical squares as neutral elements (or identity squares) :

$$\begin{array}{ccc} \bullet & \xrightarrow{1} & \bullet \\ j\downarrow & 1 & \downarrow j \\ \bullet & \xrightarrow[1]{} & \bullet \end{array}, \qquad \begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ 1\downarrow & 1 & \downarrow 1 \\ \bullet & \xrightarrow[f]{} & \bullet \end{array}.$$

Finaly, if

$$\begin{array}{ccccc}
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
\downarrow & \alpha & \downarrow & \beta & \downarrow \\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
\downarrow & \gamma & \downarrow & \delta & \downarrow \\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet,
\end{array}$$

then the following interchange law holds : $\left(\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\begin{pmatrix}\beta\\ \delta\end{pmatrix}\right) = \begin{pmatrix}(\alpha\,\beta)\\ (\gamma\,\delta)\end{pmatrix}$ and we write $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$.

If $\mathscr{D}$ and $\mathscr{E}$ are two double categories, then a *double functor* $F : \mathscr{D} \longrightarrow \mathscr{E}$ consists in two functors $HF : H\mathscr{D} \longrightarrow H\mathscr{E}$ and $VF : V\mathscr{D} \longrightarrow V\mathscr{E}$ that are compatible with all compositions.

Chapter 2

Simplicial sets

In this chapter, we introduce the notion of simplicial set, from which we'll define quasi-categories later. We also provide with important examples.

2.1 Definitions

Simplicial category. If $n \in \mathbb{N}$, denote by $[n]$ the standard total ordered $(n+1)$-element set $\{0, \dots, n\}$, and define the *simplicial category* $\mathbb{\Delta}$, the small category of all such sets, and (weakly) increasing maps between them. Some particular maps require attention :

- the *face maps* $d^i : [n] \longrightarrow [n+1]$, where $0 \leq i \leq n+1$ is the unique injective map such that $i \notin \operatorname{im} d^i$, i.e.

$$d^i\,(0 \to 1 \to \cdots \to n) = 0 \to \cdots i-1 \to i+1 \cdots \to n+1,$$

- the *degeneracy maps* $^i : [n] \longrightarrow [n-1]$, where $n > 0$, and $0 \leq i \leq n-1$, is the unique surjective map such that i is the image of two disctinct elements of $[n+1]$,

$$s^i\,(0 \to 1 \to \cdots \to n) = 0 \to \cdots i \to i \cdots \to n-1.$$

The following so-called *cosimplicial identities* give a description of the category $\mathbb{\Delta}$ in terms of generator and relations :

$$
\begin{cases}
d^j d^i = d^i d^{j-1} & \text{if } i < j \\
s^j d^i = d^i s^{j-1} & \text{if } i < j \\
s^j d^j = s^j d^{j+1} = 1 & \\
s^j d^i = d^{i-1} s^j & \text{if } i > j+1 \\
s^j s^i = s^i s^{j+1} & \text{if } i \leq j
\end{cases}
$$

Simplicial and cosimplicial objects. Let $\mathscr{C}$ be a category. A *simplicial object* X in $\mathscr{C}$ is a functor $X : \mathbb{\Delta}^{\mathrm{op}} \longrightarrow \mathscr{C}$, and we denote by $s\mathscr{C}$ the category of simplicial $\mathscr{C}$-objects, and natural transformations between them.

We note $X_n = X[n]$, and if $\alpha : [n] \longrightarrow [m]$ in $\mathbb{\Delta}$, $X_\alpha = X\alpha : X_m \longrightarrow X_n$. We also allow the following notations : $d_i = Xd^i : X_{n+1} \longrightarrow X_n$, and $s_i = Xs^i : X_n \longrightarrow X_{n+1}$. Here are some diagrams :

- in $\mathbb{\Delta}$:

$$
\begin{array}{ccccccc}
 & & & & & \xrightarrow{d^0} & \\
 & & & \xrightarrow{d^0} & & \xleftarrow{s^0} & \\
 & \xrightarrow{d^0} & & \xleftarrow{s^0} & & \xrightarrow{d^1} & \\
[0] & \xleftarrow{s^0} & [1] & \xrightarrow{d^1} & [2] & \xleftarrow{s^1} & [3]\cdots, \\
 & \xrightarrow{d^1} & & \xleftarrow{s^1} & & \xrightarrow{d^2} & \\
 & & & \xrightarrow{d^2} & & \xleftarrow{s^2} & \\
 & & & & & \xrightarrow{d^3} &
\end{array}
$$

- in $\mathscr{C}$:

$$
\begin{array}{ccccccc}
 & & & & & \xleftarrow{d_0} & \\
 & & & \xleftarrow{d_0} & & \xrightarrow{s_0} & \\
 & \xleftarrow{d_0} & & \xrightarrow{s_0} & & \xleftarrow{d_1} & \\
X_0 & \xrightarrow{s_0} & X_1 & \xleftarrow{d_1} & X_2 & \xrightarrow{s_1} & X_3\cdots. \\
 & \xleftarrow{d_1} & & \xrightarrow{s_1} & & \xleftarrow{d_2} & \\
 & & & \xleftarrow{d_2} & & \xrightarrow{s_2} & \\
 & & & & & \xleftarrow{d_3} &
\end{array}
$$

A cosimplicial $\mathscr{C}$-object Y is a functor $Y : \mathbb{\Delta} \longrightarrow \mathscr{C}$, and we denote by $c\mathscr{C}$ the category of cosimplicial $\mathscr{C}$-objects, and natural transformations between them. We allow similar abuses of notation used with simplicial objects.

The actual definition. Without surprise, a *simplicial set* X is a simplicial $\mathscr{S}et$-object, that is, a functor $X : \mathbb{\Delta}^{\mathrm{op}} \longrightarrow \mathscr{S}et$. A *n-simplex* of X is an element of X_n. A simplicial subset of X is a simplicial set $Y \in \mathrm{ob}\, s\mathscr{S}et$ such that $\forall n \in \mathbb{N}$, $Y_n \subseteq X_n$, and such that the inclusion maps define a natural transformation $X \longrightarrow Y$, i.e. a map in $s\mathscr{S}et$.

The 1-dimensional face maps $d_0, d_1 : X_1 \longrightarrow X_0$ induce an oriented multigraph which we'll note by $X|_2$.

2.2 Standard *n*-simplex

2.2.1 Definition

The *standard n-simplex* is the simplicial set $\Delta[n] = [-,[n]]_{\mathbb{\Delta}} = \mathsf{y}[n]$, where y is the Yoneda embedding[1]. Remark that the Yoneda lemma implies that for all simplicial set $X \in \mathrm{ob}\, s\mathscr{S}et$, the following (set) map is an isomorphism

$$\begin{aligned} [\Delta[n], X]_{s\mathscr{S}et} &\longrightarrow X_n \\ \phi &\longmapsto \phi_n(1_{[n]}). \end{aligned}$$

Let $0 \leq a_0 \leq \ldots \leq a_k \leq n$, and define

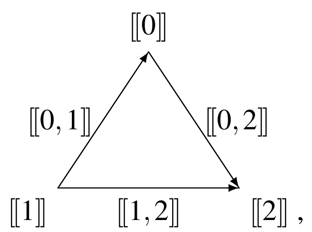
$$\begin{aligned} [\![a_0, \ldots, a_k]\!] : [k] &\longrightarrow [n] \\ k &\longmapsto a_k, \end{aligned}$$

i.e. the unique application $[k] \longrightarrow [n], k \longmapsto a_k$. For instance, take $n = 2$. Then

- $\Delta[2]_0 = [[0],[2]]_{\mathbb{\Delta}}$ can be imagined as :

$$\begin{array}{ccc} & [\![0]\!] & \\ & & \\ [\![1]\!] & & [\![2]\!] \ ; \end{array}$$

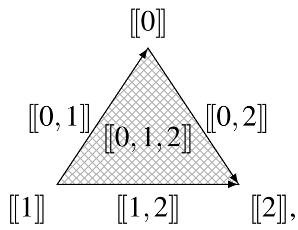
- $\Delta[2]_1 = [[1],[2]]_{\mathbb{\Delta}}$ can be imagined as :



[1] $\Delta = \mathsf{y} : \mathbb{\Delta} \longrightarrow s\mathscr{S}et$ is then a cosimplicial simplicial set !

here, we ignored degenerate maps, i.e. non injective, as they can be imagined as lines with the same ends, for instance $[\![0,0]\!] = s_0([\![0]\!])$ is identified with $[\![0]\!]$;

- $\Delta[2]_2 = [[2],[2]]_\Delta$ can be imagined as :

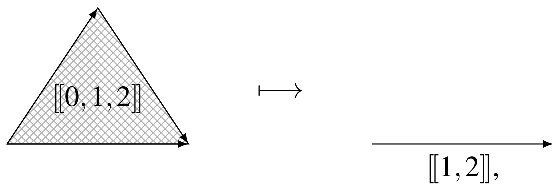


- if $m \geq 3$, $\Delta[2]_m$ consist only in degenerate maps, and may be imagined as a m-dimentional tetrahedron squashed by its vertices into a 2-dimentional tetrahedron (a triangle) ;

- here is an example of face map :

$$\begin{aligned} d_0 : \Delta[2]_2 &\longrightarrow \Delta[2]_1 \\ [a,b,c] &\longmapsto [b,c], \end{aligned}$$

  and so



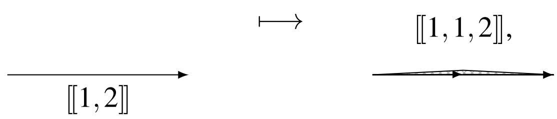
  more generaly, if $m \geq 1$, $0 \leq i \leq m+1$, then

$$\begin{aligned} d_i : \Delta[n]_m &\longrightarrow \Delta[n]_{m-1} \\ [\![a_0,\ldots,a_m]\!] &\longmapsto [\![a_0,\ldots,a_{i-1},a_{i+1},\ldots,a_m]\!] \,; \end{aligned}$$

- here is an example of degeneracy map :

$$\begin{aligned} s_1 : \Delta[2]_1 &\longrightarrow \Delta[2]_2 \\ [\![a,b]\!] &\longmapsto [\![a,b,b]\!]\,, \end{aligned}$$

  and so for instance, $[\![1,2]\!]$ is mapped to the degenerate[2] 2-simplex $[\![1,1,2]\!]$



[2] somewhat “smashed”

and more generaly, if $m \geq 0$, $0 \leq i \leq m$, then

$$\begin{aligned} s_i : \Delta[n]_m &\longrightarrow \Delta[n]_{m+1} \\ [\![a_0, \dots, a_m]\!] &\longmapsto [\![a_0, \dots, a_i, a_i, a_{i+1}, \dots, a_m]\!] . \end{aligned}$$

2.2.2 Simplicial subsets of $\Delta[n]$

Boudaries. Let $n \geq 1$. Then the *boundary of* $\Delta[n]$ is the smallest simplicial set $\partial\Delta[n] \in \mathrm{ob}\, s\mathscr{S}et$ such that $\{d_i([\![0, \dots, n]\!]) = d_i(1_{[n]}) \mid 0 \leq i \leq n\} \subseteq \partial\Delta[n]_{n-1}$. In other words :

$$\partial\Delta[n]_j = \begin{cases} \Delta[n]_j & \text{if } j < n, \\ \text{it. degens. of elems. of } \Delta[n]_k, \forall 0 \leq k \leq n-1 & \text{if } j \geq n. \end{cases}$$

More precisely, "iterated degeneracies of elements of $\Delta[n]_k$, $\forall 0 \leq k \leq n-1$" stands for the set

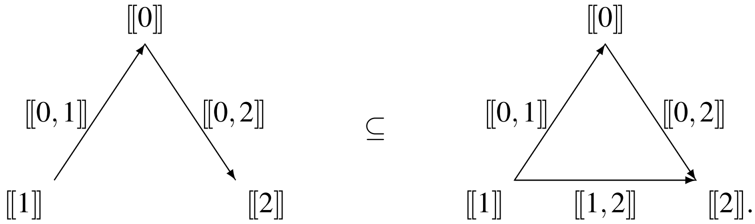
$$\partial\Delta[n]_j = \bigcup_{k=0}^{n-1} \left\{ s_{i_{j-n+1}} \cdots s_{i_0}(x) \mid x \in \Delta[n]_k, 0 \leq i_l \leq l+1 \right\} .$$

Here is a little intuition : we start at level $(n-1)$ (i.e. at $\Delta[n]_{n-1}$) with the faces of the only non degenerate n-simplex $1_{[n]} = [0, \dots, n] \in \Delta[n]_n$, i.e. $(n-1)$-simplices of the form $[\![0, \dots, \hat{i}, \dots, n]\!]$. From that, we generate the smallest simplicial set possible, that is $\partial\Delta[n]$. In a first time, we go down, applying face maps repetedly. We then go up back to $\partial\Delta[n]_{n-1}$ applying degeneracies repededly, and from here, we can already notice that the levels lower than, and at $(n-1)$ of $\Delta[n]$ and $\partial\Delta[n]$ are equal. We then go up to infinity from $\partial\Delta[n]_{n-1}$, applying degeneracies repetedly. One might wonder what would happen if we decided to do down, back to level $(n-1)$ again, applying face maps. Nothing actually, as face maps are inverse to some degeneracy map, that has been used previously. So we add no more simplex by going down from above level $(n-1)$, and we finish the job by endlessly going up.

One can find the following explicit definition :

$$\partial\Delta[n]_j = \left\{ f : [j] \longrightarrow [n] \mid \exists g : [j] \longrightarrow [n-1], \exists 0 \leq i \leq n, f = d^i g \right\} .$$

Horns. Another important simplicial subset of $\Delta[n]$ is the k-th *horn* $\Lambda^k[n]$, where $0 \leq k \leq n+1$, which is generated exactly as $\partial\Delta[n]$, except that $\Lambda^k[n]_{n-1}$ doesn't contain $d_k(1_{[n]}) = [\![0, \ldots, \hat{k}, \ldots, n]\!]$. For instance, $\Lambda^0[2]_1$ can be seen as



A horn $\Lambda^k[n]$ is said *inner* if $0 < k < n$. In a similar fashion as with $\partial\Delta[n]$, we obtain the following explicit definition :

$$\Lambda^k[n]_j = \left\{ f : [j] \longrightarrow [n] \mid \exists g : [j] \longrightarrow [n-1], \exists 0 \leq i \leq n, i \neq k, f = d^i g \right\}.$$

Chapter 3

Homotopical category

This chapter is about the adjunction $h \dashv N$ between the homocopical category functor and the nerve functor, which provides a link between simplicial sets and categories.

3.1 Nerve of a category

We can see every object $[n] \in \mathrm{ob}\,\mathbb{\Delta}$ as a (small) category, whose morphisms and composition are induced by the natural ordering, and consider the face maps and degeneracies as functors (recall that they are incresing maps). This gives rise to a subcategory of $\mathscr{C}at$ that is isomorphic to $\mathbb{\Delta}$. Let $\mathscr{C} \in \mathrm{ob}\,\mathscr{C}at$, and define its *nerve* as the simplicial set $N\mathscr{C} = [-,\mathscr{C}]_{\mathscr{C}at}|_{\mathbb{\Delta}} : \mathbb{\Delta}^{\mathrm{op}} \longrightarrow \mathscr{S}et$. In other words, a n-simplex of $N\mathscr{C}$ is a chain of composable arrows

$$x = C_0 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} C_n.$$

If $0 \leq i \leq n$, the i-th face $d_i(x) \in N\mathscr{C}_{n-1}$ of x is defined by :

$$d_i(x) = \begin{cases} C_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} C_n & \text{if } i = 0, \\ C_0 \xrightarrow{\alpha_1} \cdots C_{i-1} \xrightarrow{\alpha_{i+1}\alpha_i} C_{i+1} \cdots \xrightarrow{\alpha_n} C_n & \text{if } 1 \leq i \leq n-1, \\ C_0 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} C_{n-1} & \text{if } i = n. \end{cases}$$

and if $0 \leq i \leq n+1$, the i-th degeneracy $s_i(x) \in N\mathscr{C}_{n+1}$ of x is :

$$s_i(x) = C_0 \xrightarrow{\alpha_1} \cdots C_i \xrightarrow{1} C_i \cdots \xrightarrow{\alpha_n} C_n.$$

Remark that in particular, $N\mathscr{C}_0 = \text{ob}\,\mathscr{C}$, and $N\mathscr{C}_1 = \text{hom}\,\mathscr{C}$.

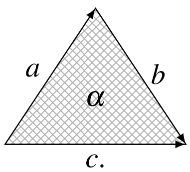
If $F : \mathscr{C} \longrightarrow \mathscr{D}$ is a functor, we can define maps $NF_n : N\mathscr{C}_n \longrightarrow N\mathscr{D}_n$ by

$$NF_n\left(C_0 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} C_n\right) = FC_0 \xrightarrow{F\alpha_1} FC_1 \xrightarrow{F\alpha_2} \cdots \xrightarrow{F\alpha_n} FC_n.$$

Moreover, those maps are compatible with the face maps and the degeneracy maps, and therefore, we have a simplicial map $NF : N\mathscr{C} \longrightarrow N\mathscr{D}$, and a functor $N : \mathscr{C}at \longrightarrow s\mathscr{S}et$, called the *nerve functor*.

3.2 Homotopicial category

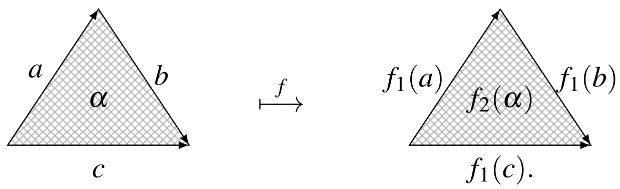
The nerve functor $N : \mathscr{C}at \longrightarrow \mathscr{S}et$ admits a left adjoint $h : s\mathscr{S}et \longrightarrow \mathscr{C}at$ which we define now. Take $X \in \text{ob}\,s\mathscr{S}et$ a simplicial set, and consider the free category $\mathscr{F} = \mathscr{F}(X|_2)$. If $A, B \in \text{ob}\,\mathscr{F}$, we define a relation on $[A,B]_{\mathscr{F}}$ as follow : $\forall ba, c : A \longrightarrow B$ (where $A \xrightarrow{a} C \xrightarrow{b} B$ for some object C), $ba \sim c$ if and only if there exists a 2-simplex $\alpha \in X_2$ of the form



Let $\simeq$ be the congruence relation on $\mathscr{F}$ generated by $\sim$, and define $hX = \mathscr{F}/\simeq$, the *homotopical category* of X. If $f : X \longrightarrow Y$ is a simplicial map, then define the functor $hf : hX \longrightarrow hY$ to be

$$\begin{aligned}(hf)_{\text{ob}} = f_0 : \text{ob}\,hX = X_0 &\longrightarrow \text{ob}\,hY = Y_0\\ (hf)_{\text{hom}} : \text{hom}\,hX &\longrightarrow \text{hom}\,hY\\ [x]_{\simeq,X} &\longmapsto [f_1(x)]_{\simeq,Y}.\end{aligned}$$

The second map is well defined. Indeed, $f : X \longrightarrow Y$ is a simplicial map, and so 2-simplices are mapped as follow :



In particular, $ba \sim c \implies f_1(b)f_1(a) \sim f_1(c), \forall a, b, c \in X_1$, and so $f_1 : X_1 \longrightarrow Y_1$ is compatible with $\sim$, and therefore with the congruence relation $\simeq$. Hence, we have a well defined functor $hf : hX \longrightarrow hY$.

Lemma 3.2.1. *Let $X \in \mathrm{ob}\, s\mathscr{S}et$ be a simplicial set, $\mathscr{C} \in \mathrm{ob}\mathscr{C}at$ be a category, and $f : X \longrightarrow N\mathscr{C}$. Then $\forall n \in \mathbb{N}^*$, f is uniquely determined by $f_n : X_n \longrightarrow N\mathscr{C}_n$.*

Proof.

- We first show that f is uniquely determined by f_1. Remark that $f_0(x) = d_0 f_1 s_0(x)$, and so f_0 is uniquely determined. Next, remark that if $x \in X_k$, then
$$f_k(x) = \left(C_0 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} C_k\right),$$
where, letting $y_i = d_0^{i-1} d_{i+1} \cdots d_k(x)$,
$$\begin{cases} \alpha_i = f_1(y_i), & 1 \leq i \leq k \\ C_0 = d_1(y_1) & \\ C_i = d_0(y_i) & 0 < i \leq k. \end{cases}$$
Intuitively, applying $d_{i+1} \cdots d_k$ and d_0^{i-1} on $x \in X_k$ "crops" the last $n-i$ and first $i-1$ 1-simplices, so that only the i-th remains. The C_is are obtained symply by applying one more face map (domain or codomain) on the y_is. Setting $y_i = d_0 \cdots d_{i-2} d_{i+1} \cdots d_k = d_0 \cdots \hat{d}_{i-1} \hat{d}_i \cdots d_k(x)$ is also correct, and crops from right to left, sparing the i-th simplex.

- Using the same "cropping" argument, we can recover f_1 starting from f_n, for any $n \geq 1$.

□

Lemma 3.2.2. *Let X and $\mathscr{C}$ be as before. Then any graph map $(f_1, f_0) : X|_2 \longrightarrow N\mathscr{C}|_2$ such that $\forall x \in X_2$, $f_1 d_1(x) = (f_1 d_0(x))(f_1 d_2(x))$, induce a uniquely determined simplicial map $f : X \longrightarrow N\mathscr{C}$.*

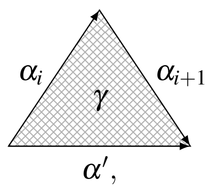
Proof. For $n \in \mathbb{N}$, define f_n as before. To check that those maps induce a natural transformation $f : X \longrightarrow N\mathscr{C}$, it is sufficent to see that it preserves the face maps and the degeneracies.

- Face maps. Let $n \geq 2$. It is clear that f_n and f_{n-1} are compatible with d_0 and d_{n+1}. Let $1 \leq i \leq n$, and $\beta \in X_n$ be such that $d_0^{i-1} d_{i+1} \cdots d_n(\beta) = \alpha_i$. On the one hand we have
$$\begin{aligned} & f_{n-1} d_i(\beta) \\ = & \left(f_0(C_0) \xrightarrow{f_1(\alpha_1)} \cdots f_0(C_{i-1}) \xrightarrow{f_1(\alpha')} f_0(C_{i+1}) \cdots \xrightarrow{f_1(\alpha_n)} f_0(C_n)\right), \end{aligned}$$

and on the other hand we have

$$\begin{aligned}
& d_i f_n(\beta) \\
&= d_i\left(f_0(C_0)\overset{f_1(\alpha_1)}{\longrightarrow}\cdots\overset{f_1(\alpha_n)}{\longrightarrow} f_0(C_n)\right) \\
&= \left(f_0(C_0)\overset{f_1(\alpha_1)}{\longrightarrow}\cdots f_0(C_{i-1})\overset{f_1(\alpha_i)f_1(\alpha_{i-1})}{\longrightarrow} f_0(C_{i+1})\cdots\overset{f_1(\alpha_n)}{\longrightarrow} f_0(C_n)\right).
\end{aligned}$$

Remark that $\gamma = d_0^{i-1}d_{i+2}\cdots d_n\beta \in X_2$ is as follow :



and by assuptions, $f_1(\alpha') = f_1(\alpha_{i+1})f_1(\alpha_i)$.

- Degeneracies. Let $n \geq 2$, $0 \leq i \leq n$, and $\beta \in X_n$ be as before. Then

$$\begin{aligned}
& f_{n+1}s_i(\beta) \\
&= \left(f_0(C_0)\overset{f_1(\alpha_1)}{\longrightarrow}\cdots f_0(C_i)\overset{f_0(s_0C_i)}{\longrightarrow} f_0(C_i)\cdots\overset{f_1(\alpha_n)}{\longrightarrow} f_0(C_n)\right) \\
&= \left(f_0(C_0)\overset{f_1(\alpha_1)}{\longrightarrow}\cdots f_0(C_i)\overset{s_0f_1(C_i)}{\longrightarrow} f_0(C_i)\cdots\overset{f_1(\alpha_n)}{\longrightarrow} f_0(C_n)\right) \\
&= s_i\left(f_0(C_0)\overset{f_1(\alpha_1)}{\longrightarrow}\cdots\overset{f_1(\alpha_n)}{\longrightarrow} f_0(C_n)\right) \\
&= s_i f_n(\beta).
\end{aligned}$$

□

Theorem 3.2.3. *We have an adjunction $h \dashv N$.*

Proof. We use the hom-set caracterisation of adjunctions. Let $X \in \mathrm{ob}\, s\mathscr{S}et$, and $\mathscr{C} \in \mathrm{ob}\mathscr{C}at$. Define $H_{\mathscr{C}at}$ to be the set of functors $F : \mathscr{F}X|_2 \longrightarrow \mathscr{C}$ such that for all 2-simplex $x \in X_2$ we have $F(d_2(x)d_0(x)) = Fd_1(x)$. Define $H_{\mathscr{S}et}$ to be the set of maps $f_1 : X_1 \longrightarrow N\mathscr{C}_1$ such that $\forall x \in X_2$ we have $f_1d_1(x) = f_1d_0(x)f_1d_2(x)$. Consider the chain bijections

$$[hX,\mathscr{C}]_{\mathscr{C}at} \longleftrightarrow H_{\mathscr{C}at} \longleftrightarrow H_{\mathscr{S}et} \longleftrightarrow [X,N\mathscr{C}]_{s\mathscr{S}et},$$

where the first two are obviously defined, and the last one is defined using the previous two lemmas. Remark that they all are natural bijections in X and $\mathscr{C}$, and so is their composite. □

Lemma 3.2.4. *We have $hN \cong 1$. Consequently, h is surjective and full, and N is an embedding.*

Proof. Let $\mathscr{C} \in \mathrm{ob}\mathscr{C}at$. Recall the process that constructs the homotopical category of $N\mathscr{C}$. We first consider the free category spanned by the underlying graph of $\mathscr{C}$: $\mathscr{F} = \mathscr{F}(\hom\mathscr{C} \rightrightarrows \mathrm{ob}\mathscr{C})$, which we then quotient by a congruence relation. But one can check that a morphism $m_k \cdots m_1$ in $\mathscr{F}$, i.e. a word of morphisms of $\mathscr{C}$ with matching codomains and domains from a letter to the next, is precisely identified with the composite in $\mathscr{C}$ of the letters : $m_k \cdots m_1 \simeq m_k \circ \cdots \circ m_1$. Therefore, $hN\mathscr{C} = \mathscr{C}$. Next, it can be seen from the definition that $hNF = F$, for all functor $F \in \hom\mathscr{C}at$. Thereby, $hN \cong 1$. □

Chapter 4

Quasi-categories

We now define the notion of quasi category, and provide a different construction for the homotopical category of a quasi category. From here, the nerve functor N will prove to be a 2-embedding $\mathscr{C}at \longrightarrow q\mathscr{C}at_2$, i.e. that quasi categories is indeed a generalisation of categories.

4.1 Definition

A simplicial set $X \in \mathrm{ob}\, s\mathscr{S}et$ is a *quasi-category* if it has the right (not necessarily unique) lifting property with respect to all inner horn inclusion $\Lambda^k[n] \hookrightarrow \Delta[n]$, $\forall n \geq 2$, i.e.

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{\ \forall\ } & X \\ \big\downarrow & \nearrow_{\exists} & \\ \Delta[n]. & & \end{array}$$

Denote by $q\mathscr{C}at$ the full subcategory of $s\mathscr{S}et$ spanned by quasi-categories.

4.2 A new look at the adjunction $h \vdash N$

Nerve. The functor h admits a much simplier description if restricted to $q\mathscr{C}at$, which we'll make explicit later. We first show that the nerve functor N corestricts to a functor $N : \mathscr{C}at \longrightarrow q\mathscr{C}at$.

Lemma 4.2.1. *Let $\mathscr{C} \in \mathrm{ob}\mathscr{C}at$ be a category. Then its nerve $N\mathscr{C}$ is a quasi-category. Moreover, the lift is unique.*

Proof. Let $n \geq 2$, $1 \leq k \leq n-1$, and $f : \Lambda^k[n] \longrightarrow N\mathscr{C}$ be a simplicial map. Then there exists $C_0, \dots, C_n \in \mathrm{ob}\mathscr{C}$, $\alpha_i : C_{i-1} \longrightarrow C_i$ such that

$$f_{n-1}(d_i(1_{[n]})) = \begin{cases} C_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} C_n & \text{if } i = 0, \\ C_0 \xrightarrow{\alpha_1} \cdots C_{i-1} \xrightarrow{\alpha_{i+1}\alpha_i} C_{i+1} \cdots \xrightarrow{\alpha_n} C_n & \text{if } 1 \leq i \leq n-1, i \neq k, \\ C_0 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} C_{n-1} & \text{if } i = n. \end{cases}$$

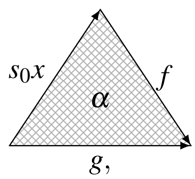
Define $\tilde{f} : \Delta[n] \longrightarrow N\mathscr{C}$ by extending f in the following way :

$$\tilde{f}_n(1_{[n]}) = \left(C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} C_n \right).$$

It is well defined, and thus $N\mathscr{C}$ is a quasi-category. If $\tilde{f}'$ is another extension of f, then for $1 \leq i \leq n$ we have $\tilde{f}_1(d_0 \cdots \hat{d}_{i-1}\hat{d}_i \cdots d_n(1_{[n]})) = \alpha_i = \tilde{f}'_1(d_0 \cdots \hat{d}_{i-1}\hat{d}_i \cdots d_n(1_{[n]}))$ which implies $\tilde{f}_n = \tilde{f}'_n$, and so $\tilde{f} = \tilde{f}'$, by lemma 3.2.2. □

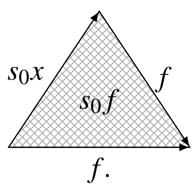
Homotopicial category. If $X \in \mathrm{ob}\, q\mathscr{C}at$, we define a category $\dot{h}X$ as follow.

- $\mathrm{ob}\, \dot{h}X = X_0$.
- $\forall x, y \in \mathrm{ob}\, \dot{h}X$, denote by $H(x,y) = \{f \in X_1 \mid d_0 f = y, d_1 f = x\}$, define an equivalence relation on $H(x,y)$ by $f \dot{\simeq} g$ if and only if $\exists \alpha \in X_2$ such that $d_0\alpha = f$, $d_1\alpha = g$, $d_2\alpha = s_0 x$, i.e.

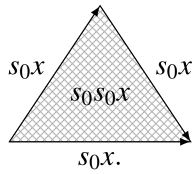


  and define define $[x,y]_{\dot{h}X} = H(x,y)/\dot{\simeq}$. We have to check that $\dot{\simeq}$ is indeed an equivalence relation.

  – Reflexivity. Take $f \in H(x,y)$, and consider $s_0 f$:



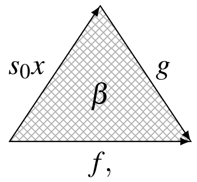
– Symmetry. Take $f,g \in H(x,y)$ such that $f \dot{\simeq} g$. Consider $\alpha \in X_2$ the simplex that gives $f \dot{\simeq} g$ (defined above), and $s_0 s_0 x$:



There is a well defined simplicial map

$$\begin{aligned} \phi : \Lambda^2[3] &\longrightarrow X \\ [\![1,2,3]\!] = d_0(1_{[n]}) &\longmapsto \alpha \\ [\![0,2,3]\!] = d_1(1_{[n]}) &\longmapsto s_0 f \\ [\![0,1,2]\!] = d_3(1_{[n]}) &\longmapsto s_0 s_0 x, \end{aligned}$$

which lifts trough $\Delta[3]$. Remark that $\beta = \phi([\![0,1,3]\!])$ is as follow :

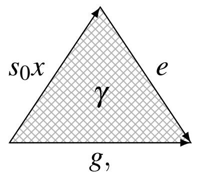


and so $g \dot{\simeq} f$.

– Transitivity. Let $e,f,g \in H(x,y)$ be such that $e \dot{\simeq} f \dot{\simeq} g$, and let $\alpha, \beta \in X_2$ be the associated 2-simplicies respectively. As before, there is a well defined simplicial map

$$\begin{aligned} \phi : \Lambda^1[3] &\longrightarrow X \\ [\![1,2,3]\!] = d_0(1_{[n]}) &\longmapsto \alpha \\ [\![0,1,3]\!] = d_2(1_{[n]}) &\longmapsto \beta \\ [\![0,1,2]\!] = d_3(1_{[n]}) &\longmapsto s_0 s_0 x, \end{aligned}$$

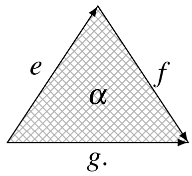
which lifts trough $\Delta[3]$. We have that $\gamma = \phi([\![0,2,3]\!])$ is as follow :



and so $e \dot{\simeq} g$.

- If $x \in \operatorname{ob} \dot{h}X$, the identity morphism is given by $[s_0 x]_{\dot{\simeq}}$,

- If $e, f, g \in X_1$, then $[f]_{\simeq}[e]_{\simeq} = [g]_{\simeq}$ if and only if $\exists \alpha \in X_2$ such that $d_0\alpha = e$, $d_1\alpha = g$, and $d_2\alpha = f$, i.e.



If $f : X \longrightarrow Y$ is a simplicial map, we can define a functor $\dot{h}f : \dot{h}X \longrightarrow \dot{h}Y$ is a very similar way that with h. We then have a functor $\dot{h} : q\mathscr{C}at \longrightarrow \mathscr{C}at$.

Adjunction. In a very similar way than with the "classical" homotopical category functor h, we show that we have an adjunction $\dot{h} \dashv N : q\mathscr{C}at \rightleftarrows \mathscr{C}at$. Consequently, we have the following result :

Proposition 4.2.2. *There is a natural isomorphism $h \cong \dot{h} : q\mathscr{C}at \longrightarrow \mathscr{C}at$.*

Proof. We have isomorphisms

$$[h-_1, -_2]_{\mathscr{C}at} \cong [-_1, N-_2]_{q\mathscr{C}at} \cong [\dot{h}-_1, -_2]_{\mathscr{C}at}$$

that are natural in both $-_1$ and $-_2$. By the Yoneda lemma, $h \cong \dot{h}$ naturally. □

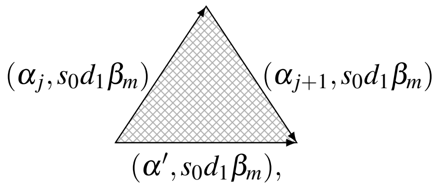
4.3 The 2-category of quasi-categories

Lemma 4.3.1. *The functor $h : s\mathscr{S}et \longrightarrow \mathscr{C}at$ is monoidal.*

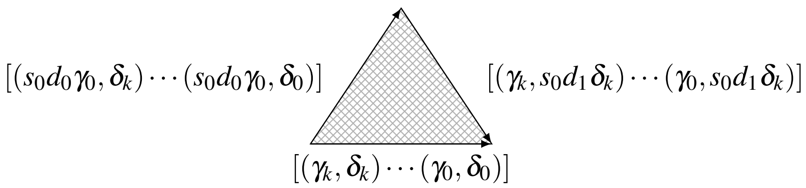
Proof. Let $X, Y \in \text{ob}\, s\mathscr{S}et$ be two simplicial sets, we show that $h(X \times Y) \cong hX \times hY$ using the universal property of the product. First of all, we can endow $h(X \times Y)$ with the obvious projection maps. Let $\mathscr{C} \in \text{ob}\mathscr{C}at$ be a category, and $F : \mathscr{C} \longrightarrow hX$, $F' : \mathscr{C} \longrightarrow hY$ be two functors. Define

$$\begin{aligned} G : \mathscr{C} &\longrightarrow h(X \times Y) \\ C &\longmapsto (FC, F'C) \\ f &\longmapsto [(\alpha_n, s_0d_0\beta_m)\cdots(\alpha_0, s_0d_0\beta_m)(s_0d_1\alpha_0, \beta_m)\cdots(s_0d_1\alpha_0, \beta_0)]_{\simeq, X\times Y}, \end{aligned}$$

where $Ff = [\alpha_n \cdots \alpha_0]_{\simeq, X}$ and $F'f = [\beta_m \cdots \beta_0]_{\simeq, Y}$. We check that G is well defined for $\sim$. Suppose that $\alpha' \in X_1$ is such that $\alpha_{j+1}\alpha_j \sim \alpha'$, for some $0 \leq j < n$, and let $H \in X_2$ be the associated 2-simplex. Remark that $(H, s_0s_0d_1\beta_m) \in (X \times Y)_2$ is of the following form :



and hence $(\alpha_{j+1}, s_0 d_1 \beta_m)(\alpha_j, s_0 d_1 \beta_m) \sim (\alpha', s_0 d_1 \beta_m)$. It comes that G is compatible with $\sim$ in its first variable. The same reasoning goes for the second, and as $\simeq$ are generated by $\sim$, it results that G is well defined. Clearly, it commutes with the projection maps. To see that it is unique on the gound of objects is trivial. Suppose that $G' : \mathscr{C} \longrightarrow h(X \times Y)$ is another functor that commutes with the projections. Let $f \in \hom \mathscr{C}$, and $[(\gamma_k, \delta_k) \cdots (\gamma_0, \delta_0)] = G' f$. Remark that



$= [(\gamma_k, \delta_k) \cdots (\gamma_0, \delta_0)]$.

Hence, G' can we written in the same way than G, making the two necessarily equal. □

We show that $s\mathscr{S}et$ is cartesian closed.

- It is easy to see that $\Delta[0]$ is terminal.
- We can define a product in $s\mathscr{S}et$ using the product in $\mathscr{S}et$ pointwise, i.e. if $X, Y \in \mathrm{ob}\, s\mathscr{S}et$, then $(X \times Y)_n = X_n \times Y_n$, and the faces and degeneracies are given by functoriality of $\times$ in $\mathscr{S}et$.
- If $X, Y \in \mathrm{ob}\, s\mathscr{S}et$, we define the internal hom as $(Y^X)_n = [X \times \Delta[n], Y]_{s\mathscr{S}et}$, where the face maps are given by

$$d_i = (1 \times d^i_*)^* : [X \times \Delta[n], Y]_{s\mathscr{S}et} \longrightarrow [X \times \Delta[n-1], Y]_{s\mathscr{S}et},$$

  where $d^i_* : \Delta[n-1] = [-, [n-1]]_{\Delta} \longrightarrow \Delta[n] = [-, [n]]_{\Delta}$ is the postcomposition by $d^i : [n-1] \longrightarrow [n]$, and the degeneracies s_i are defined similarily.

Theorem 4.3.2. *The simplicial set Y^X defined above is indeed an exponential object.*

Proof. Take $A, B, C \in \mathrm{ob}\, s\mathscr{S}et$. We show that $[A, C^B]_{s\mathscr{S}et} \cong [A \times B, C]_{s\mathscr{S}et}$. Take $f : A \longrightarrow C^B$ a simplicial map. We have maps $f_n : A_n \longrightarrow [B \times \Delta[n], C]$. Define

$$\begin{aligned} \hat{f}_n : (A \times B)_n = A_n \times B_n &\longrightarrow C_n \\ (\alpha, \beta) &\longmapsto f_n(\alpha)_n(\beta, 1_{[n]}). \end{aligned}$$

We show that the $\hat{f}_n$ induce a simplicial map $\hat{f} : A \times B \longrightarrow C$. Suppose $n \geq 1$, take $0 \leq i \leq n+1$, and $(\alpha, \beta) \in (A \times B)_n$. Then

$$\begin{aligned} \hat{f}_{n-1}(d_i\alpha, d_i\beta) &= f_{n-1}(d_i\alpha)_{n-1}(d_i\beta, 1_{[n-1]}) \\ &= ((1 \times d_*^i)^* f_n(\alpha))_{n-1}(d_i\beta, 1_{[n-1]}) \\ &= f_n(\alpha)_{n-1}(d_i\beta, d_*^i 1_{[n-1]}) \\ &= f_n(\alpha)_{n-1}(d_i\beta, d_i 1_{[n]}) \\ &= d_i f_n(\alpha)_n(\beta, 1_{[n]}). \end{aligned}$$

Using a similar reasonning for the degeneracies s_i, we obtain that $\hat{f} : A \times B \longrightarrow C$ is indeed a implicial map. Take now $g : A \times B \longrightarrow C$, and define

$$\begin{aligned} g'(\alpha, n)_k : B_k \times \Delta[n]_k &\longrightarrow C_k \qquad \text{for } \alpha \in A_n \\ (\beta, \delta) &\longmapsto g_k(A_\delta \alpha, \beta), \\ \check{g}_n : A_n &\longrightarrow [B \times \Delta[n], C]_{s\mathscr{S}et} \\ \alpha &\longmapsto g'(\alpha, n). \end{aligned}$$

Then the $(\hat{-})$ and the $(\check{-})$ constructions are mutually inverse, which show that the $\check{g}_n$ give rise to a well defined simplicial map $\check{g}$, and that $[A, C^B]_{s\mathscr{S}et} \cong [A \times B, C]_{s\mathscr{S}et}$. □

Denote by $q\mathscr{C}at_\infty$ the category $q\mathscr{C}at$ seen as a $s\mathscr{S}et$-enriched category, and by $q\mathscr{C}at_2 = h_* q\mathscr{C}at_\infty$ the category $q\mathscr{C}at_\infty$ seen as a $\mathscr{C}at$-enriched category (i.e. a 2-category) trough the functor h. Also, denote by $\mathscr{C}at_2 = N\mathscr{C}at$ the full subcategory of $q\mathscr{C}at_2$ spanned by nerves of categories.

Lemma 4.3.3. *Let $X, Y \iota\, \mathrm{ob}\, q\mathscr{C}at$. Then the simplicial* hom Y^X *is a quasi category.*

Proof. This proof requiers knowledge about model categories, and can be found in [2]. □

Theorem 4.3.4. *The category $q\mathscr{C}at_2$ is cartesian closed as a 2-category.*

Proof.
- The above terminal simplicial set $\Delta[0]$ is a quasi-category, as the only map $\Lambda^k[n] \xrightarrow{!} \Delta[0]$ with $0 < k < n$ lifts trough $\Lambda^k[n] \hookrightarrow \Delta[n]$ (in a unique way moreover). It is a 2-categorical terminal object, as $\forall X \in \mathrm{ob}\, q\mathscr{C}at_2$, we have $[X, \Delta[0]]_{q\mathscr{C}at_2} = h[X, \Delta[0]]_{s\mathscr{S}et} \cong h\Delta[0] \cong \mathbb{1}$, the terminal category.

- We show that the product of two quasi-categories $X_1, X_2 \in \operatorname{ob} q\mathscr{C}at_2$ is a quasi-category. Take $f : \Lambda^k[n] \longrightarrow X_1 \times X_2$, where $0 < k < n$, and consider $\operatorname{proj}_i f : \Lambda^k[n] \longrightarrow X_i$. We have a lift $g_i : \Delta[n] \longrightarrow X_i$ of $\operatorname{proj}_i f$ trough the inclusion $\iota : \Lambda^k[n] \hookrightarrow \Delta[n]$. By the universal property of the product, we have a map $g = \langle g_1, g_2 \rangle : \Delta[n] \longrightarrow X_1 \times X_2$ such that $f = g\iota$. Moreover, the product $\times$ from $s\mathscr{S}et$ is a 2-categrical product, as $\forall Y \in \operatorname{ob} q\mathscr{C}at_2$

$$\begin{aligned}
[Y, X_1 \times X_2]_{q\mathscr{C}at_2} &= h[Y, X_1 \times X_2]_{s\mathscr{S}et} \\
&\cong h([Y, X_1]_{s\mathscr{S}et} \times [Y, X_2]_{s\mathscr{S}et}) \\
&\cong h[Y, X_1]_{s\mathscr{S}et} \times h[Y, X_2]_{s\mathscr{S}et} \qquad \text{as } h \text{ is monoidal} \\
&= [Y, X_1]_{q\mathscr{C}at_2} \times [Y, X_2]_{q\mathscr{C}at_2}.
\end{aligned}$$

- Let $Y, Z \in \operatorname{ob} q\mathscr{C}at_2$, and define the internal hom as Z^Y, the same as defined above. It is a 2-categorical exponential object, as $\forall X \in \operatorname{ob} q\mathscr{C}at_2$ we have

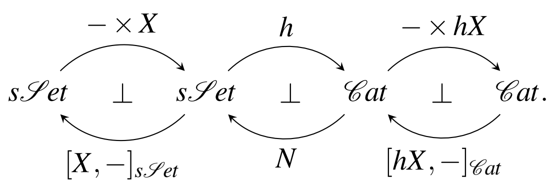
$$\begin{aligned}
[X, Z^Y]_{q\mathscr{C}at_2} &= h[X, Z^Y]_{s\mathscr{S}et} \\
&\cong h[X \times Y, Z]_{s\mathscr{S}et} \\
&= [X \times Y, Z]_{q\mathscr{C}at_2}.
\end{aligned}$$

□

Lemma 4.3.5. *There is a natural isomorphism :*

$$[-_1, N-_2]_{s\mathscr{S}et} \cong N[h-_1, -_2]_{\mathscr{C}at} : s\mathscr{S}et^{\mathrm{op}} \times \mathscr{C}at \longrightarrow s\mathscr{S}et.$$

Proof. Take $X \in \operatorname{ob} s\mathscr{S}et$. We have adjunctions



Recall that h preserve products. Hence

$$[X, N-]_{s\mathscr{S}et} \vdash h(- \times X) \cong h - \times hX \dashv N[hX, -],$$

and so $[-_1, N-_2]_{s\mathscr{S}et} \cong N[h-_1, -_2]$ naturally as required. □

Proposition 4.3.6. *We have $\mathscr{C}at \cong \mathscr{C}at_2$ as 2-categories.*

Proof. We already know that N is an embedding, and so $\mathscr{C}at \cong \mathscr{C}at_2$ as categories. Recall that $hN \cong 1$. Let $\mathscr{C}, \mathscr{D} \in \operatorname{ob} \mathscr{C}at$. Then by previous result, $N[\mathscr{C}, \mathscr{D}]_{\mathscr{C}at} \cong [N\mathscr{C}, N\mathscr{D}]_{s\mathscr{S}et}$ in $s\mathscr{S}et$. Applying h gives us the required isomorphism of hom-categories $[\mathscr{C}, \mathscr{D}]_{\mathscr{C}at} \cong [N\mathscr{C}, N\mathscr{D}]_{q\mathscr{C}at_2} = [N\mathscr{C}, N\mathscr{D}]_{\mathscr{C}at_2}$. □

Chapter 5

Adjunctions

This chapter brings the central notion of adjunctions to $q\mathscr{C}at_2$ using 2-categories and double categories. We moreover give a characterisation in term of absolute right lifting diagram.

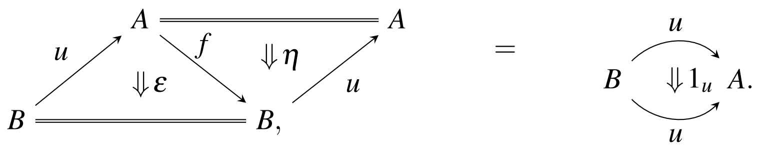
5.1 Definition

Let $\mathscr{K}$ be a 2-category, $A, B \in \mathrm{ob}\,\mathscr{K}$. An *adjunction* in $\mathscr{K}$ consists in two antiparallel 1-cells $f : A \rightleftarrows B : u$, and two 2-cells $\eta : 1 \longrightarrow uf$, $\varepsilon : fu \longrightarrow 1$ such that the following triangle identities holds :

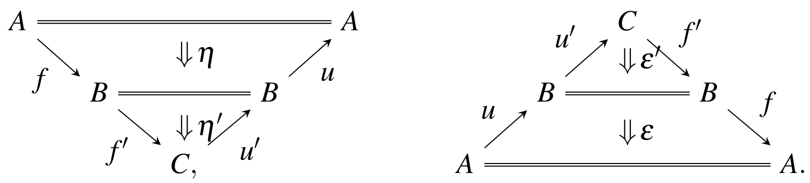
$$\begin{tikzcd} f \arrow[rr, "1"] \arrow[dr, "f\eta"'] & & f \\ & fuf, \arrow[ur, "\varepsilon f"'] & \end{tikzcd} \qquad \begin{tikzcd} u \arrow[rr, "1"] \arrow[dr, "\eta u"'] & & u \\ & ufu, \arrow[ur, "u\varepsilon"'] & \end{tikzcd}$$

or, in term of pasting diagrams,

$$\begin{tikzcd} A \arrow[rr, equal] \arrow[dr, "f"'] & \Downarrow \eta & A \arrow[dr, "f"] & \\ & B \arrow[ur, "u"] \arrow[rr, equal] & \Downarrow \varepsilon & B, \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] & B, \end{tikzcd} \ (\Downarrow 1_f)$$



We then note $f \dashv u : A \longrightarrow B$ by omitting η and ε. We can compose adjunctions in the following way : if $A \xrightarrow{f \dashv u} B \xrightarrow{f' \dashv u'} C$, then define $f'f \dashv uu' : A \longrightarrow C$ by

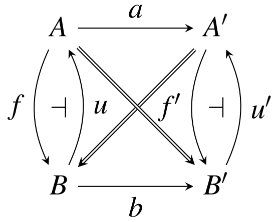


Together with the obvious identity adjunction $1,1 : 1 \dashv 1 : A \longrightarrow A$, we obtain a category $\mathscr{K}_{\dashv}$ whose objects are those of $\mathscr{K}$, and morphisms are adjunctions in $\mathscr{K}$.

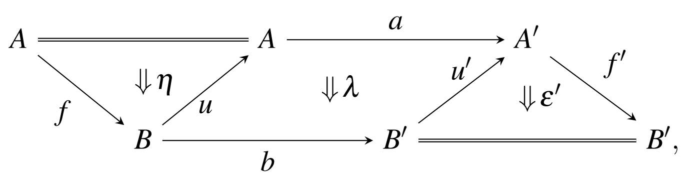
Proposition 5.1.1. *Let $\mathscr{K}$ and $\mathscr{L}$ be two 2-categories, and $F : \mathscr{K} \longrightarrow \mathscr{L}$ be a 2-functor. Then an adjunction $\eta, \varepsilon : f \dashv u$ in $\mathscr{K}$ gives rise to an adjunction $F\eta, F\varepsilon : Ff \dashv Fu$ in $\mathscr{L}$. Hence, we have a functor $F_{\vdash} : \mathscr{K}_{\vdash} \longrightarrow \mathscr{L}_{\vdash}$.*

Proof. This can be verified by applying F on the triangle identities of $\eta, \varepsilon : f \dashv u$. □

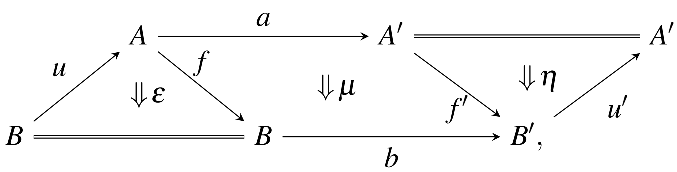
Lemma 5.1.2. *Let $f \dashv u : A \longrightarrow B$ and $f' \dashv u' : A' \longrightarrow B'$ be two adjunctions, and $a : A \longrightarrow A'$, $b : B \longrightarrow B'$ be two morphisms. We have a bijection $[au, u'b]_{[A,B']} \cong [f'a, bf]_{[A,B']}$.*



Proof. If $\lambda : au \longrightarrow u'b$, define $\Phi(\lambda)$ as



and if $\mu : f'a \longrightarrow bf$, define $\Psi(\mu)$ as



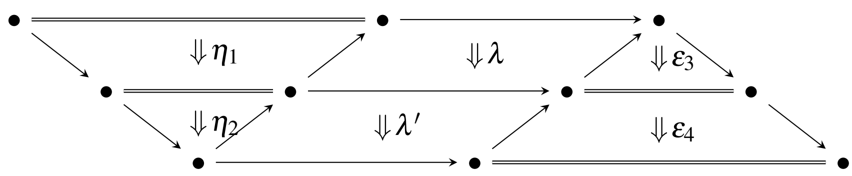
Using the hypothesis of adjunctions $f \dashv u$ and $f' \dashv u'$, we obtain that Φ and Ψ are mutually inverse. $\square$

If $\lambda \in [au, u'b]_{[A,B']}$ and $\mu \in [f'a, bf]_{[A,B']}$ are such that $\Phi(\lambda) = \mu$ (or equivalently $\lambda = \Psi(\mu)$), then those 2-cells are said *mates* under the adjunctions $f \dashv u$ and $f' \dashv u'$. Define the double category $\mathscr{K}_1$ as $H\mathscr{K}_1 = \mathscr{K}$, $V\mathscr{K}_1 = \mathscr{K}_\dashv$, and having as squares the 2-cells of the form $\lambda : au \longrightarrow u'b$, using the previous notations. Define $\mathscr{K}_2$ the same way, except that the squares are given by 2-cells of the form $\mu : f'a \longrightarrow bf$.

Lemma 5.1.3. *By extending the definition of Φ and Ψ to any pair of adjunction connected by an adequate pair of morphisms, we obtain two double functors Φ : $\mathscr{K}_1 \rightleftarrows \mathscr{K}_2 : \Psi$ that are mutually inverse.*

Proof. We already know that Φ and Ψ are identities on the objects, vertical and horizontal morphisms, and mutually inverse on squares. Clearly, they preserve all kind of identities. It remains to show that they are compatible with compositions. It is sufficient to show it for Φ. The vertical composition are preserved by definition of compositions on $\mathscr{K}_\dashv$:

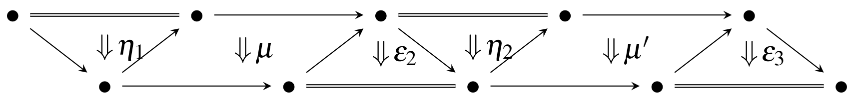
$$\Phi\left(\begin{pmatrix}\lambda \\ \lambda'\end{pmatrix}\right) =$$

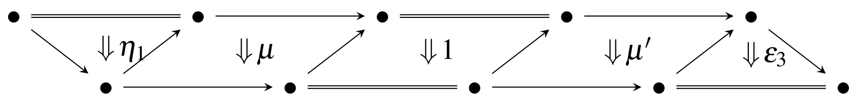


$$= \begin{pmatrix}\Phi(\lambda) \\ \Phi(\lambda')\end{pmatrix},$$

whereas horizontal composition are as well, by definition of an adjunction :

$$(\Phi(\mu)\,\Phi(\mu')) =$$



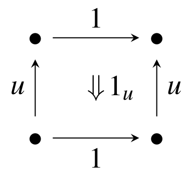
$=$ 

$$= (\Phi(\mu)\,\Phi(\mu')).$$

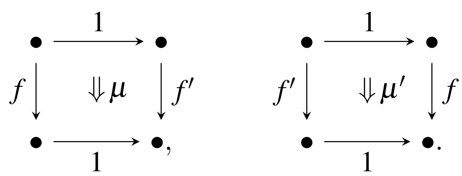
□

Theorem 5.1.4. *Left and right adjoints are unique up to isomorphism, should they exist.*

Proof. Suppose $f \dashv u$ and $f' \dashv u$. Let the mates of



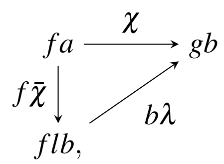
under the adjunctions $f \dashv u$ and $f' \dashv u$ be respectively



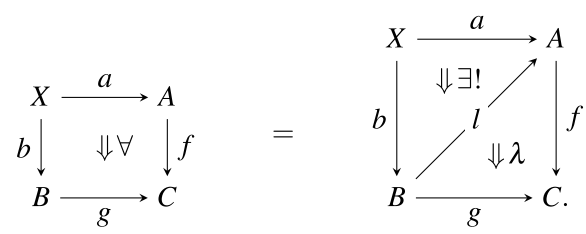
Then $\Psi((\mu\,\mu')) = (\Psi(\mu)\,\Psi(\mu')) = (1_u\,1_u) = 1_u$, and so $(\mu\,\mu') = 1_f$ as Ψ is an isomorphism $\mathscr{K}_2 \xrightarrow{\cong} \mathscr{K}_1$. Similarly, $(\mu'\,\mu) = 1_{f'}$, and so $f \cong f'$. The unicity of right adjoints is obtained in a very similar way. □

5.2 Absolute right lifting

Let $\mathscr{K}$ be a 2-category. An *absolute right lifting diagram* is a 2-cell $\lambda : fl \longrightarrow g$ such that $\forall \chi : fa \longrightarrow gb$, $\exists! \bar{\chi} : a \longrightarrow lb$ such that

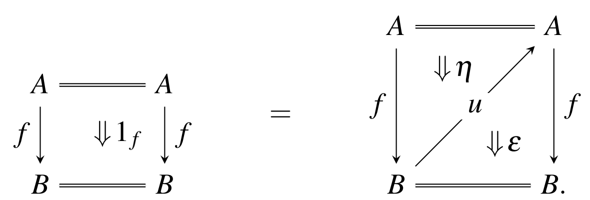


or, in term of parting diagrams :

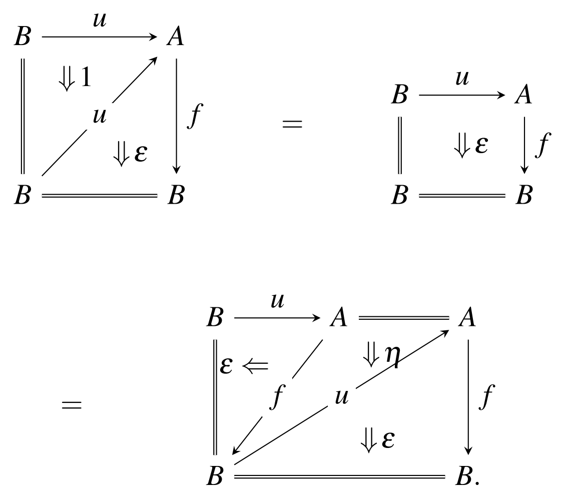


Theorem 5.2.1. *Let $\varepsilon : fu \longrightarrow 1$ be a 2-cell. Then it is an absolute right lifting diagram if and only if it is the counit of an adjunction $f \dashv u$.*

Proof. • Suppose that ε is an absolute right lifting. Define $\eta = \overline{1_f}$:



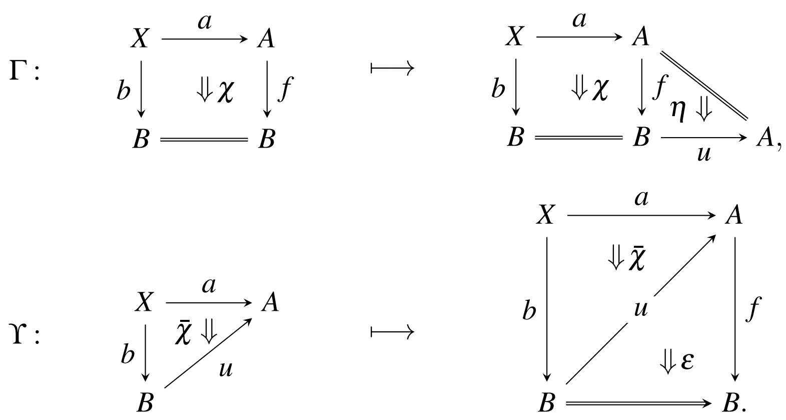
The first triangle identity is already verified. Consider



By the uniqueness of the lift, the second triangular identity is verified.

• Suppose that we have an adjunction $\eta, \varepsilon : f \dashv u$, and let take $X \in \mathrm{ob}\mathscr{K}$,

$a : X \longrightarrow A$, and $b : X \longrightarrow B$. Define $\Gamma : [fa, b] \rightleftarrows [a, ub] : \Upsilon$ by



By the triangle identities, we have that Γ and Υ are mutually inverse. Hence, any 2-cell χ as above admit a unique right lift $\Gamma(\chi)$ trough ε.

□

Chapter 6

Limits

Recall that $\forall A \in \mathrm{ob}\, s\mathscr{S}et$, we have an adjunction $A \times - \dashv (-)^A$. If $B \in \mathrm{ob}\, s\mathscr{S}et$, denote by $c : B \longrightarrow B^A$ the map associated to the projection $\pi : A \times B \longrightarrow B$ in the hom-set natural isomorphism $[A \times B, B] \cong [B, B^A]$.

Remark that $\Delta[0]$ is a terminal object in $s\mathscr{S}et$. Hence it is a quasi category, and a neutral element for the product. In a similar manner as before, we obtain natural isomorphisms $[J,X]_{q\mathscr{C}at} \cong [\Delta[0], X^J]_{s\mathscr{S}et}$, where $J, X \in \mathrm{ob}\, q\mathscr{C}at$. If $f : J \longrightarrow X$, we identify it with it associated morphism $f : \Delta[0] \longrightarrow X^J$.

Let $d : J \longrightarrow X$ be a diagram (i.e. a morphism). An element $l : \Delta[0] \longrightarrow X$ is a *limit* of d if there is an absolute right lifting diagram as follows :

$$\begin{array}{ccc} & & X \\ & \overset{l}{\nearrow} \;\; \Downarrow \lambda & \big\downarrow c \\ \Delta[0] & \xrightarrow[d]{\qquad} & X^J. \end{array}$$

Proposition 6.0.2. *Limits are unique up to isomorphism, should they exist.*

Proof. If $m : \Delta[0] \longrightarrow X$ is another limit of d, with absolute right lifting μ, then there exists a 2-cell $\bar{\mu}$ such that

$$\begin{array}{ccc} \Delta[0] & \xrightarrow{m} & X \\ \Vert & \Downarrow \mu & \downarrow c \\ \Delta[0] & \xrightarrow[d]{} & X^J \end{array} \quad = \quad \begin{array}{ccc} \Delta[0] & \xrightarrow{m} & X \\ \Vert & \Downarrow \bar{\mu} \nearrow_{l} \Downarrow \lambda & \downarrow c \\ \Delta[0] & \xrightarrow[d]{} & X^J. \end{array}$$

Similarily, there exists a 2-cell $\bar{\lambda}$ which factors λ through μ. By the universal property of the absolute right lifting, we have that $\bar{\lambda}$ and $\bar{\mu}$ are mutually inverse, which shows that $l \cong m$. □

More generally, let $k : K \longrightarrow X^J$ be a morphism. We say that X admits all limits of the family of diagrams k if there exists an absolute right lifting

$$\begin{array}{ccc} & & X \\ & {}^{\lim}\nearrow \;\Downarrow \lambda & \downarrow c \\ K & \xrightarrow[k]{} & X^J. \end{array}$$

Proposition 6.0.3. *The quasi category X admits all limits of the family of diagrams k if and only if every diagram $d : \Delta[0] \longrightarrow X^J$ that factors trough k admit a limit.*

Proof. See proposition 5.2.10 p. 59 in [2]. □

Corollary 6.0.4. *If all diagram $d : \Delta[0] \longrightarrow X$, admit a limit, then we have an adjunction $c \dashv \lim : X \longrightarrow X^J$.*

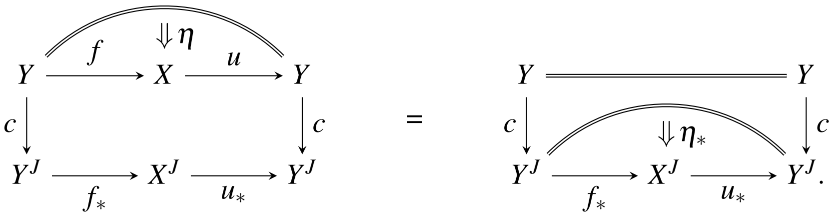
Proof. Consider the family of diagrams $1 : X^J \longrightarrow X^J$. By assumption and the previous result, X admits all limits of 1, i.e. we have an absolute right lifting

$$\begin{array}{ccc} & & X \\ & {}^{\lim}\nearrow \;\Downarrow \lambda & \downarrow c \\ X^J & = & X^J. \end{array}$$

Therefore, by theorem 5.2.1, $c \dashv \lim$. □

Lemma 6.0.5. *Let $\eta, \varepsilon : f \dashv u : Y \longrightarrow X$ be an adjunction. By proposition 5.1.1 and the fact that $(-)^J$ as a 2-functor, we have an adjunction $\eta_*, \varepsilon_* : f_* \dashv u_* : Y^J \longrightarrow A^J$.*

We claim that the following 2*-cells are equal :*



Proof. Recall that $\eta_* : 1 \longrightarrow (uf)_*$ is such that $(\eta_*)_t = \eta 1_t, \forall t \in Y^J$. The above 2-cell equality can be stated with a commutative diagram :

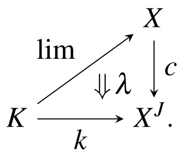
$$\begin{array}{ccc} c & \xrightarrow{c\eta} & cuf \\ \| & & \| \\ c & \xrightarrow{\eta_* c} & (uf)_* c. \end{array}$$

Take $y \in Y$, a simplex of arbitrary dimension. Then we have

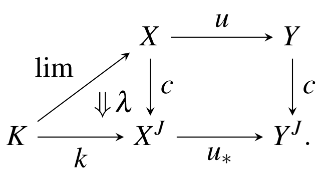
$$\begin{array}{ccc} c(y) & \xrightarrow{(c\eta)_y = c\eta_y} & c(uf(y)) \\ \| & & \| \\ c(y) & \xrightarrow{(\eta_* c)_y = (\eta_*)_{c(y)} = \eta 1_{c(y)}} & ufc(y), \end{array}$$

which commutes, for all y. □

Theorem 6.0.6. *Right adjoints preserve limits. More explicitely, let $f \dashv u : Y \longrightarrow X$ be an adjunction, and consider the following limit :*

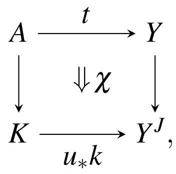


Then the following 2*-cell is an absolute right lifting :*

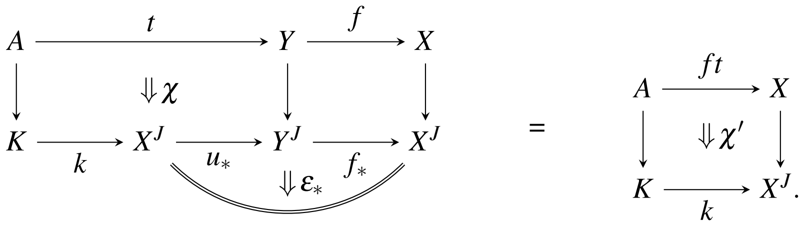


(6.1)

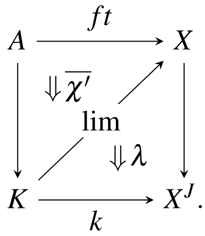
Proof. Start with a two cell



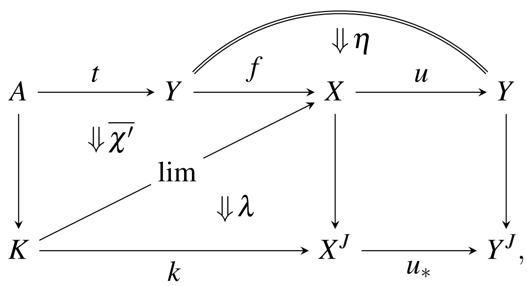
and consider



By the universal property of the absolute right lifting, we have a unique 2-cell $\overline{\chi'}$:



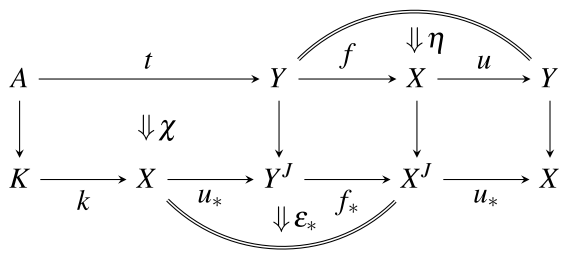
which we extend into



(6.2)

and it remains to show that the latter 2-cell equals χ. We have

(6.2) =



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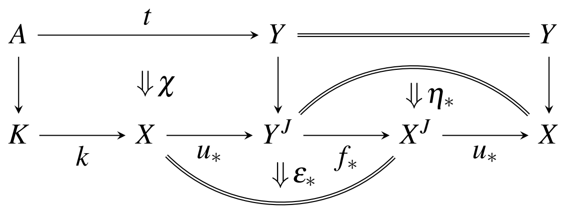

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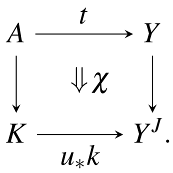
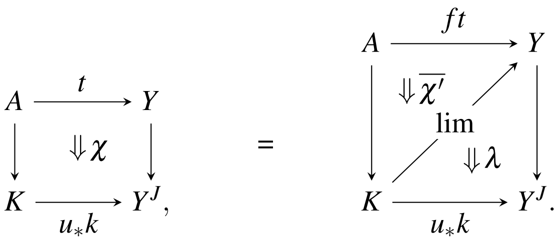
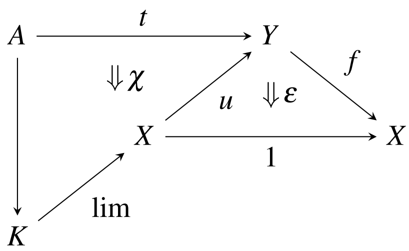


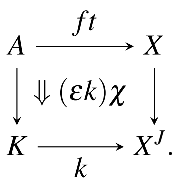
Conversey, start with a lift



Then remark that



is a lift of



One could show that the two constructions shown above are inverse to each other, and so the lift in diagram 6.2 is unique. It follows diagram 6.1 is an absolute right lifting. □

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