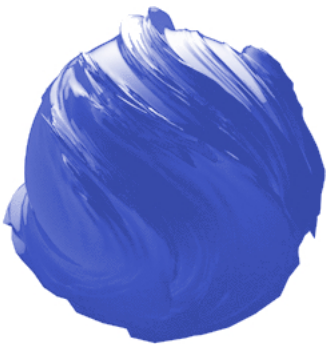
Finite dimentional algebras

Unofficial notes from the course of prof. Jaques Thévenaz
TEXed by Cédric Ho Thanh

Spring 2015



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Definition. This document is unofficial lecture notes from the course *Finite dimentional algebras* given by prof. Jaques Thévenaz at EPFL during the spring semester 2015.

Corollary. *This document is provided as is, with potentially numerous typos and errors, without warranty of any kind.*

Chapter 1

Algebras and modules

1.1 Algebras

For all this course, the letter $\mathbb{K}$ will denote a (commutative) field.

Definition 1.1.1 ($\mathbb{K}$-algebra)**.** A $\mathbb{K}$-*algebra* is a quadruple $(A,+,\cdot,*)$ where

1. $(A,+,\cdot)$ is a ring with unit 1_A,
2. $(A,+,*)$ is a $\mathbb{K}$-vector space,
3. $\forall\lambda\in\mathbb{K},\forall a,b\in A$ we have

$$(\lambda * a)\cdot b = \lambda * (a\cdot b) = a\cdot(\lambda * b).$$

In order to simplify notations, we'll denote $\lambda a = \lambda * a$.

Remark 1.1.2. Since a $\mathbb{K}$-algebra A is also a $\mathbb{K}$-vector space, we call the *dimension* of A its dimension as a $\mathbb{K}$-vector space.

Example 1.1.3. $\mathbb{C}$ is a $\mathbb{C}$-algebra of dimension 1, and a $\mathbb{R}$-algebra of dimension 2.

Definition 1.1.4 ($\mathbb{K}$-algebra, alternate definition)**.** . A $\mathbb{K}$-algebra is a quadruple $(A,+,\cdot,\phi)$ where :

1. $(A,+,\cdot)$ is a ring with unit 1_A,
2. $\phi:\mathbb{K}\longrightarrow Z(A)$ is a ring homomorphism.

Proposition 1.1.5. *The two definitions of* $\mathbb{K}$*-algebra are equivalent.*

Proof. Take A a $\mathbb{K}$-algebra in the sense of definition 1.1.1, and define

$$\begin{aligned}\phi:\mathbb{K}&\longrightarrow Z(A)\\ \lambda&\longmapsto \lambda * 1_A.\end{aligned}$$

Condition 3 of definition 1.1.1 states that the above morphism is well defined. Conversely, take B a $\mathbb{K}$-algebra in the sense of definition 1.1.4, and define the $*$ operation by

$$\begin{aligned} * : \mathbb{K} \times B &\longrightarrow B \\ (\lambda, b) &\longmapsto \lambda * b = \phi(\lambda) \cdot b. \end{aligned}$$

Since ϕ takes its image in $Z(B)$, the above operation satisfies condition 2 and 3 of definition 1.1.1. □

Remark 1.1.6. The morphism $\phi : \mathbb{K} \longrightarrow Z(A)$ is always injective, unless $A = 0$.

Examples 1.1.7.

1. $\mathbb{K}$ is a $\mathbb{K}$-algebra of dimension 1 ;

2. $M_n(\mathbb{K})$ with the morhism

$$\begin{aligned} \phi : \mathbb{K} &\longrightarrow M_n(\mathbb{K}) \\ \lambda &\longmapsto \lambda I_n \end{aligned}$$

   is a $\mathbb{K}$-algebra of dimension n^2 ;

3. If $(A, +, \cdot, *)$ is a $\mathbb{K}$ algebra, the *opposite algebra* $(A^{\mathrm{op}}, +, \odot, *)$ is defined as follows :

   (a) $(A, +, *) = (A^{\mathrm{op}}, +, *)$ as $\mathbb{K}$-vector spaces,

   (b) $\forall a, b \in A^{\mathrm{op}}$, $a \odot b = b \cdot a$;

4. If G is a group, the *group algebra* $\mathbb{K}G$ is a $\mathbb{K}$-algebra of dimension $|G|$;

5. The previous example also works if G is a monoid, and the result is called a *monoid algebra* ;

6. A *quiver* $Q = (V, E, \varepsilon)$ is a finite directed (multi-)graph :

   (a) V is a finite set of *vertices*,

   (b) E is a finite set of *directed edges*,

   (c)

$$\begin{aligned} \varepsilon : E &\longrightarrow V \times V \\ e &\longmapsto (\varepsilon_0(e), \varepsilon_1(e)), \end{aligned}$$

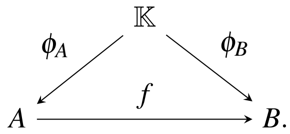
   where $\varepsilon_0(e)$ and $\varepsilon_1(e)$ are the *origin* and *target* of e respectively.

   For every vertex $v \in V$, we define a path l_v of length 0 from v to v, which is called a *lazy path*. Let $\mathbb{K}Q$ be the $\mathbb{K}$-vector space with basis the set of paths of Q. The multiplication is induced by the concatenation of paths :

$$(e_1, \dots, e_n) \cdot (f_1, \dots, f_m) = \begin{cases} (e_1, \dots, e_n, f_1, \dots, f_m) & \text{if } \varepsilon_1(e_n) = \varepsilon_0(f_1) \\ 0 & \text{otherwise,} \end{cases}$$

   and the neutral element is given by $\sum_{v \in V} l_v$.

Definition 1.1.8 (Morphism of algebras)**.** Let A and B be two $\mathbb{K}$-algebras. A *morphism of algebras* $f : A \longrightarrow B$ is a $\mathbb{K}$-linear ring morphism. An algebra *isomorphism* is a bijective algebra morphism. Starting with definition 1.1.4, f is a algebra morphism if it is a morphism of ring such that the following diagram commutes :



1.2 Modules

1.2.1 Basic definitions

Definition 1.2.1 (R-Module)**.** Let R be a ring (with unit). A *left R-module* is a triple $(M, +, *)$ where

1. $(M, +)$ is an abelian group,
2. $* : R \times M \longrightarrow M$ is such that $\forall r, s \in R, \forall m, m' \in M$

$$\begin{aligned}(r+s)*m &= r*m+s*m\\ r*(m+m') &= r*m+r*m'\\ (r\cdot s)*m &= r*(s*m)\\ 1_R*m &= m.\end{aligned}$$

Again, in order to simplify notation, we'll denote $rm = r*m$. A *right R-module* is defined in a similar way, but with $* : M \times R \longrightarrow M$. If M is a left R-module, we emphasize this structure with the notation ${}_RM$. Similarily, if M is a right R-module, we note M_R.

Properties 1.2.2.

$$\begin{aligned}r0_M &= 0_M\\ 0_Rm &= 0_M\\ r(-m) &= -(rm) = (-r)m.\end{aligned}$$

Remark 1.2.3. Since algebras are ring, we have a definition of module M over an $\mathbb{K}$-algebra A. Moreover, in this case, M is also a $\mathbb{K}$-vector space (by restriction of scalar along $\phi : \mathbb{K} \longrightarrow A$).

Definitions 1.2.4. Let M be an R-module.

1. A subgroup $N \leq M$ that is stable under $r*-$, $\forall r \in R$ is called a *submodule* of M ;

2. A submodule of ${}_RR$ is a left ideal of R, whereas a submodule of R_R is a right ideal ;

3. Let $m_1, \ldots, m_n \in M$. An *R-linear combination* of thos elements is an element of M of the form $\sum_{i=1}^n r_i m_i$, for some $r_i \in R$;

4. The module M is said *finitely generated* if there exist a finite subset $X \subseteq M$ such that every element of M is a R-linear combination of elements in X ;

5. If $N, L \leq M$, we define their *sum* by

$$N + L = \{n + l \mid n \in N, l \in L\};$$

   The sum is *direct* if $N \cap L = 0$, and we note $N + L = N \oplus L$;

6. A submodule $N \leq M$ is called a *direct summand* if there exists $L \leq M$ such that $M = N \oplus L$;

Definition 1.2.5 (Morphism of modules)**.** Let M and N be two left R-modules. A *morphim of modules* (of *R-linear map*) $f : M \longrightarrow N$ is a group homomorphism such that the following diagram commutes :

$$\begin{CD} R \times M @>{\mathrm{id} \times f}>> R \times N \\ @V{*}VV @VV{*}V \\ M @>{f}>> N. \end{CD}$$

We denote by $\mathrm{Hom}_R(M, N)$ the set of all R-linear maps from M to N. The definition of morphism of right R-modules is obtained similarily.

1.2.2 Quotients

Let M be a R-module, and $L \subseteq M$ be a submodule. Then L is a normal subgroup of M, and the quotient group M/L is defined. Also, there is a quotient homomorphism $\pi : M \longrightarrow M/L$. Since L is a submodule, M/L acquires a structure of R-module :

$$r * \bar{m} = \overline{r * m}, \qquad \forall m \in M, r \in R.$$

Moreover : $\pi : M \longrightarrow M/L$ becomes a morphism of R-modules.

Theorem 1.2.6 (Universal property of the quotient module)**.** *For any morphism of R-modules $\phi : M \longrightarrow N$ such that $\phi(L) = \{0\}$, there exists a unique homomorphism of R-modules $\bar{\phi} : M/L \longrightarrow N$, such that the following diagram commutes :*

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ {\scriptstyle \pi}\downarrow & \nearrow_{\exists \bar{\phi}} & \\ M/L. & & \end{array}$$

Proof. Follows easily from the universal property of the quotient group. □

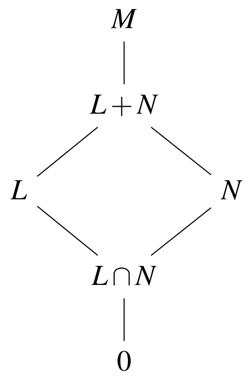
Exercise 1.2.7. $\pi : M \longrightarrow M/L$ induces a bijection :

$$\{X \text{ submodule of } M \mid L \subseteq X\} \overset{\cong}{\longleftrightarrow} \{Y \text{ submodule of } M/L\}.$$

Theorem 1.2.8 (Isomorphism theorems)**.**

1. *Let $\phi : M \longrightarrow N$ be a homomorphism of R-modules. Then $\bar{\phi} : M/\ker\phi \overset{\cong}{\longrightarrow} \operatorname{im}\phi$ is an isomorphism.*

2. *Let N and L be submodules of M. Then the inclusion $L \hookrightarrow L+N$ induces an isomorphism of R-modules $L/(L\cap N) \overset{\cong}{\longrightarrow} (L+N)/N$.*



3. *Let L and N be submodules of M such that $L \subseteq M$. ThenWe have an isomorphism $M/N \overset{\cong}{\longrightarrow} (M/L)/(N/L)$.*

1.2.3 Exact sequences

Definition 1.2.9 (Exact sequence)**.** A sequence $L \overset{\phi}{\to} M \overset{\psi}{\to} N$ of two homomorphism of R-modules is said *exact* if $\operatorname{im}\phi = \ker\psi$.

Properties 1.2.10.

1. *$0 \to M \overset{\psi}{\to} N$ is exact if and only if ψ is injective.*

2. *$L \overset{\phi}{\to} M \to 0$ is exact if and only if ϕ is surjective.*

3. *Let L be a submodule of M, then the following sequence is exact : $0 \to L \hookrightarrow M \overset{\pi}{\twoheadrightarrow} M/L \to 0$.*

4. *More generaly, a sequence $0 \to L \overset{\phi}{\to} M \overset{\psi}{\to} L \to 0$ is called a* short exact sequence *if it is exact for every two consecutive pair of homomorphism :*

   - *ϕ is injective,*
   - *$\operatorname{im}\phi = \ker\psi$, and so ψ induced an isomorphism $M/\operatorname{im}\phi \overset{\cong}{\longrightarrow} N$,*

- *ψ is surjective.*

Definition 1.2.11. 1. A *section* of a homomorphism of R-modules $\phi : M \longrightarrow N$ is a homomorphism $\sigma : N \longrightarrow M$ such that $\phi\sigma = \mathrm{id}_N$. In that case, ϕ is surjective and σ is injective.

2. A *retraction* of ϕ is a homomorphism $\rho : N \longrightarrow M$ such that $\rho\phi = \mathrm{id}_M$. In that case, ϕ is injective, and ρ is surjective.

Proposition 1.2.12. *Let $0 \to L \overset{\alpha}{\to} M \overset{\pi}{\to} L \to 0$ be a short exact sequence of R-modules. The following are equivalent :*

1. *π has a section,*
2. *α has a retraction,*
3. *$\operatorname{im}\alpha$ is a direct summand of M.*

Proof.

1. $\Longrightarrow$ 2. Assume that π has a section $\sigma : N \longrightarrow M$, and define $\varepsilon = \mathrm{id}_M - \sigma\pi : M \longrightarrow M$. Clearly, $\operatorname{im}\varepsilon \subseteq \ker\pi = \operatorname{im}\alpha$. Using $\alpha^{-1} : \operatorname{im}\alpha \longrightarrow L$, we obtain $\rho = \alpha^{-1}\varepsilon : M \longrightarrow L$. Then

$$\begin{aligned}\rho\alpha &= \alpha^{-1}\varepsilon\alpha \\ &= \alpha^{-1}(\alpha - \sigma \underbrace{\pi\alpha}_{=0}) \\ &= \mathrm{id}_L .\end{aligned}$$

2. $\Longrightarrow$ 3. Assume that α has a retraction $\rho : M \longrightarrow L$. Let $Q = \ker\rho$. Then $M = Q \oplus \operatorname{im}\alpha$. Indeed, take $m \in M$ and write $m = \underbrace{\alpha\rho(m)}_{\in \operatorname{im}\alpha} + \underbrace{(m - \alpha\rho(m))}_{\in\ker\rho}$. So $M = Q + \operatorname{im}\alpha$. Let $m \in Q \cap \operatorname{im}\alpha$. Then $\exists l \in L$ suh that $\alpha(l) = m$, and so $l = \rho\alpha(l) = \rho(m) = 0$. So $l = 0$, and $m = 0$. Hence $Q \cap \operatorname{im}\alpha = \{0\}$.

3. $\Longrightarrow$ 1. Assume that $M = Q \oplus \operatorname{im}\alpha$. We first prove that $\pi|_Q : Q \longrightarrow L$ is an isomorphism. $\ker\pi|_Q = Q \cap \ker\pi = Q \cap \operatorname{im}\alpha = \{0\}$. Let $n \in N$. Since π is surjective, $\exists m \in M$ such that $\pi(m) = n$. Then $m = q + \alpha(l)$, for some $q \in Q$ and $l \in L$. However, $\pi(q + \alpha(l)) = \pi(q)$, and so $\pi|_Q$ is surjective. So it is an isomorphism. Finaly, $\sigma = \pi|_Q^{-1} : N \longrightarrow M$ is a section of π.

□

Definition 1.2.13 (Split exact sequence)**.** Where one (and hence all) of the above condition is satisfied, the given short exact sequence is said *split*.

1.2.4 Free modules

Definition 1.2.14 (Free module)**.** An R-module F is called *free* if it has a basis, i.e. a R-generating R-linearly independent subset $B \subseteq F$.

Lemma 1.2.15. *Let F be an R-module. Then F is finitely generated free if and only if $\exists n \in \mathbb{N}$ such that $F \cong R^n$.*

Proposition 1.2.16.

1. *Any R-module is isomorphic to a quotient of a free module.*

2. *Any finitely generated R-module is isomorphic to a quotient of a finitely generated free module.*

Proof. Let M be an R-module, and $G = \{g_i \mid i \in I\}$ be a set of generators of M (such a set exists : take the whole M for instance). Let F be a free module with basis $B = \{b_i \mid i \in I\}$, and define

$$\begin{aligned} \phi : F &\longrightarrow M \\ b_i &\longmapsto g_i, \end{aligned}$$

extended by R-linearity. Then $M \cong F/\ker\phi$. □

1.3 Projective modules

Proposition 1.3.1. *Let R be a ring, and let P be a left R-module. The following are equivalent :*

1. *P is isomorphic the a direct summand of a free module ;*

2. *for every surjective homomorphism π and any ϕ, there exists a lift as follows :*

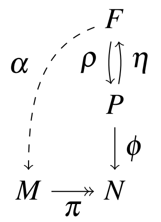
$$\begin{array}{ccc} & {\scriptstyle \exists\tilde{\phi}} & P \\ & \swarrow & \downarrow{\scriptstyle \phi} \\ L & \overset{\pi}{\twoheadrightarrow} & M; \end{array}$$

3. *for any surjective homomorphism $\phi : M \longrightarrow P$, there exists a section $\sigma : P \longrightarrow M$;*

4. *any short exact sequence of the form $0 \to L \to M \to P \to 0$ splits.*

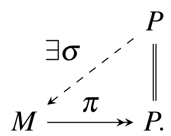
Proof.

1. $\Longrightarrow$ 2. by assumption, $P \cong P'$, and $P' \oplus Q = F$ free. Let $\eta : P \xrightarrow{\cong} P' \hookrightarrow F$, and $\rho \twoheadrightarrow P' \xrightarrow{\cong} P$.



F has a basis $G = \{g_i \mid i \in I\}$. Let m_i be a preimage of $\phi\rho(g_i)$ by π, i.e. $\pi(m_i) = \phi\rho(g_i)$. As F is free, there exists a map $\alpha : F \longrightarrow M$ such that $\alpha(g_i) = m_i$. Define $\tilde{\phi} = \alpha\eta : P \longrightarrow M$. Then $\pi\tilde{\phi} = \pi\alpha\eta = \phi\rho\eta = \phi$.

2. $\Longrightarrow$ 3. Consider



3. $\Longrightarrow$ 4. The homomorphism $M \longrightarrow P$ in the short exact sequence is surjective, and so admits a section.

4. $\Longrightarrow$ 1. Recall that P is a quotient of a free module. So we have an exact sequence $0 \to \ker\pi \to F \xrightarrow{\pi} P \to 0$. It splits by assumption, and so $F \cong P \oplus \ker\pi$.

□

Definition 1.3.2 (Projective module)**.** A module satisfying one (and hence all) of the above condition is called *projective*.

Chapter 2

Semisimplicity

2.1 Definitions

Definition 2.1.1 (Simple module)**.** Let R be a ring, and S be a left R-module. It is said *simple* if it is not trivial, and if it has no submodule other than 0 and S.

Lemma 2.1.2. *S is simple if and only if S is isomorphic to R/I (as modules), where I is a maximal left ideal of R.*

Proof. $\Longleftarrow$ We have $S \cong R/I$. A submodule of S corresponds to a submodule $I \leq M \leq R$. Hence S is simple.

$\Longrightarrow$ Let S be simple. In particular, $S \neq 0$ and $\exists s \in S \setminus \{0\}$. Since R is a free R-module with basis 1, there exists a homomorphism $\pi : R \longrightarrow S$ such that $\pi(1) = s$. We have $0 \neq \operatorname{im} \pi \leq S$, and as S is simple, $\operatorname{im} \pi = S$. By the first isomorphism theorem, $S \cong R/\ker \pi$. By simplicity of S, $\ker \pi$ is maximal.

□

Lemma 2.1.3 (Schur, general case)**.** *1. Let S be a simple left R-module. Then* $\operatorname{End}_R S$ *as an R-module is a division ring.*

2. If S and T are two nonisomorphic simple R-modules, then $\operatorname{Hom}_R(S,T) = 0$.

Proof. 1. Let $\phi \in \operatorname{End}_R S$. Remark that $\ker \phi, \operatorname{im} \phi \leq S$, and so either

(a) $\operatorname{im} \phi = 0$, $\ker \phi = S$, in which case $\phi = 0$;

(b) $\ker \phi = 0$, $\operatorname{im} \phi = S$, in which case ϕ is an automorphism, hence invertible.

2. Similarily, a homomorphism $\phi : S \longrightarrow T$ is either 0 or an isomorphism.

□

Lemma 2.1.4 (The *real* lemma of Schur)**.** *Let A be a finite dimensional $\mathbb{K}$-algebra, and let S be a simple A-module.*

1. *$K \hookrightarrow \mathrm{End}_A S \hookrightarrow \mathrm{End}_{\mathbb{K}} S = \mathrm{Mat}_n(\mathbb{K})$, where $n = \dim_{\mathbb{K}} S$;*

2. *if $\mathbb{K}$ is algebraically closed, then $\mathbb{K} \cong \mathrm{End}_A S$.*

Proof. 1. If S is simple, we have $S \cong A/I$, for some (maximal) left ideal I, and so S is finitely generated. So S is finite dimensional (cf exercise 1, sheet 2) as a $\mathbb{K}$-vector space. Therefore, $\mathrm{End}_{\mathbb{K}} S \leq \mathrm{Mat}_n(\mathbb{K})$, where $n = \dim_{\mathbb{K}} S$. Obviously, $\mathrm{End}_A S$ is a subalgebra of $\mathrm{End}_{\mathbb{K}} S$, and we have an embedding $\mathbb{K} \hookrightarrow \mathrm{End}_A S : \lambda \longmapsto \lambda\, \mathrm{id}_S$.

2. Assume $\mathbb{K}$ algebraically closed, and let $\phi \in \mathrm{End}_A S$. Let $\mathbb{K}[X]$ be the polynomial ring in one variable X, and define $\pi : \mathbb{K}[X] \longrightarrow \mathbb{K}[\phi] \leq \mathrm{End}_A S : X \longmapsto \phi$. Then π is a $\mathbb{K}$-algebra map. By the general case of Schur lemma, we know that $\mathrm{End}_A S$ is a division algebra, and so $\rho\psi \neq 0$ whenever $\rho, \psi \neq 0$, for $\rho, \psi \in \mathrm{End}_A S$. Therefore, the commutative subalgebra $\mathbb{K}[\phi]$ is a domain. Since $\mathbb{K}[X]$ is a PID, $\ker \pi$ is generated by a single polynomial f which can be chosen monic (π can't be injective, because $\mathbb{K}[X]$ is infinite dimensional, and $\dim \mathrm{End}_A S \leq n^2$), i.e. f is the minimal polynomial of ϕ. So $\mathbb{K}[\phi] \cong \mathbb{K}[X]/\langle f \rangle$. But this is a domain, so f is irreducible. Since $\mathbb{K}$ is algebraically closed, $f(X) = X - \lambda$, for a $\lambda \in \mathbb{K}$. It follows that $\mathbb{K}[\phi] \cong \mathbb{K}[X]/\langle X - \lambda \rangle \cong \mathbb{K}$, and so $\phi = \lambda\, \mathrm{id}_S$. Hence, the map $\mathbb{K} \longrightarrow \mathrm{End}_S A : \lambda \longmapsto \lambda\, \mathrm{id}_S$ is an isomorphism.

□

Proposition 2.1.5. *Let A be a finite dimensional $\mathbb{K}$-algebra, and let M a be finitely generated left A-module. The following are equivalent :*

1. *M is a sum of simple submodules, i.e. $M = \sum_{i \in I} S_i$, where $S_i \leq M$ is simple ;*

2. *M is a direct sum of finitely many simple submodules, i.e. $M = \bigoplus_{j \in J} S_j$, where $S_j \leq M$ is simple, and J finite ;*

3. *every submodule $N \leq M$ is a direct summand of M.*

Proof.

1. $\Longrightarrow$ 2. Define $X = \{J \subseteq I \mid \sum_{j \in J} S_j \text{ is direct}\}$. Such a $J \in X$ must be finite, as $\infty > \dim_{\mathbb{K}} M \geq \dim_{\mathbb{K}} \sum_{j \in J} S_j = \dim_{\mathbb{K}} \bigoplus_{j \in J} S_j = \sum_{j \in J} \dim_{\mathbb{K}} S_j \geq \sharp J$. Take a maximal element J in X. Then we claim that $S_i \subseteq \bigoplus_{j \in J} S_j$, $\forall i \in I$. Assume $i \notin J$. If $S_i \not\subseteq \bigoplus_{j \in J} S_j$, then $S_i \cap \bigoplus_{j \in J} S_j$ is a proper submodule of S_i, and so it is 0. So the sum $S_i + \bigoplus_{j \in J} S_j$ is direct, which is absurd by maximality of J. So $M = \sum_{i \in I} S_i \subseteq \bigoplus_{j \in J} S_j$, and finally, $M = \bigoplus_{j \in J} S_j$.

2. $\Longrightarrow$ 3. We have $M = \bigoplus_{j \in J} S_j$, and let $N \leq M$. Suppose $N \neq M$, and define $Y = \{L \subseteq J \mid N + \bigoplus_{l \in L} S_l \text{ is direct}\}$. Let L be maximal in Y. We claim that $N \oplus \bigoplus_{l \in L} S_l = M$. Let $j \in J \setminus L$, then $N + (S_j \oplus \bigoplus_{l \in L} S_l)$ is not direct by maximality of L. So $S_j \cap N$ is not 0, so it is S_j by simplicity. For every $j \in J$, $S_j \subseteq N \oplus \bigoplus_{l \in L} S_l$, and so $N \oplus \bigoplus_{l \in L} S_l = M$.

3. $\Longrightarrow$ 1. Let $T = \sum_{S \subseteq M, S \text{ simple}} S$. It is a submodule of M, and so by assumption, $M = T \oplus U$, for some $U \leq M$. If $U \neq 0$, then we choose a nonzero submodule $V \leq U$ of minimal $\mathbb{K}$-dimension. Then V must be simple by minimality, and then we get a direct sum $T \oplus V \leq T \oplus U = M$. On the other hand, $V \leq T$ by definition of T. So $V = U = 0$, and $T = M$.

□

Definition 2.1.6. An A-module M satisfying one (and hence all) of the above condition is said *semisimple*.

Remark 2.1.7. The results of proposition 2.1.5 holds for any ring R. The proof uses Zorn's lemma.

Definition 2.1.8. 1. A finitely generated A-module M is called *semisimple isotypic* if M is the direct sum of isomorphic simple modules. We have $M \cong S^{\oplus k}$, for S a simple module, and we call S its *type*. Lemma 2.1.9 shows that this definition is well founded.

2. If M is semisimple, then we can group all isomorphic simple summands in its decomposition as a direct sum of simple modules, and obtain the following :

$$M = \underbrace{(S_{1,1} \oplus \cdots \oplus S_{1,n_1})}_{\text{all isomorphic}} \oplus \cdots \oplus \underbrace{(S_{m,1} \oplus \cdots \oplus S_{m,n_m})}_{\text{all isomorphic}}$$

where $S_{i,j} \ncong S_{k,l}$ whenever $i \neq k$. Then $S_i = \bigoplus_{j=1}^{n_i} S_{i,j}$ is called an *isotypic component* of M, of type $S_{i,1} \cong \cdots \cong S_{i,n_i}$.

Lemma 2.1.9. *1. Any simple submodule of a semisimple isotypic module N of type S is isomorphic to S.*

2. If $M = \bigoplus_S M_S$ is a decomposition of M into isotypic components, with M_S of type S, then any simple submodule of M isomorphic to S is contained in M_S.

3. M_S only depends on the type S, not on the chosen simple decomposition of M. Explicitely, $M_S = \bigoplus_{T \leq M, T \cong S} T$.

Proof. 1. $N = \bigoplus_{i=1}^{n} S_i$, where $S_i \cong S$. Let $T \leq N$ be simple. Then $\text{proj}_{S_i} T$ is not always 0, otherwise we would have $T = 0$. Take j such that $0 \neq \text{proj}_{S_j} T \leq S_j$. Because S_j is simple, we have $\text{proj}_{S_j} T = S_j$. Moreover, since T is simple, $\ker(\text{proj}_{S_j} : T \longrightarrow S_j) = 0$, and so $T \cong S_j$.

2. Let $T \leq M$ be a simple submodule isomorphic to S. There exists $U \leq M$ simple such that $\text{proj}_{M_U} T \neq 0$. However, $\text{proj}_{M_U} T$ is a simple submodule of M_U isomorphic to T. So by the previous point, $U \cong T \cong S$. The same reasinong applies to show that $\text{proj}_{M_V} T = 0$, whevever $V \ncong S$. So $T \leq M_S$.

3. Obvious from previous points.

□

Example 2.1.10. Let $A = \mathbb{K}$. A theorem in linear algebra states that every A-module ($\mathbb{K}$-vector space) has a basis. So every A-module is semisimple isotypic of type $\mathbb{K}$.

Definition 2.1.11. Let A be a finite dimensional $\mathbb{K}$-algebra.

1. A is called *semisimpe algebra* if ${}_AA$ is a semisimple module.
2. A is called *simple* if ${}_AA$ is semisimple isotypic.

Proposition 2.1.12. *The following are equivalent :*

1. *A is semisimple ;*
2. *every finitely generated left A-module is projective ;*
3. *every finitely generated left A-module is semisimple.*

Proof.

1. $\Longrightarrow$ 2. Let M be a finitely generated left A-module. Then $\exists n \in \mathbb{N}$ such that $M \cong F/N$, where $F = A^{\oplus n}$ is free, and $N \leq F$. Moreover, F is semisimple, and so N is a direct summand of F, i.e. $F = N \oplus Q$, for some submodule Q. So $M \cong Q$ is projective.

2. $\Longrightarrow$ 3. Let N be a submodule of M. We have a short exact sequence $0 \to N \to M \to M/N \to 0$. By assumption, M/N is projective, so the previous exact sequence splits, and $M \cong N \oplus M/N$. So N is a direct summand of M, and M is semisimple.

3. $\Longrightarrow$ 1. Remark that A is a finitely generated A-module.

□

Exercise 2.1.13. If A is semisimple, let $A \cong \bigoplus_{i=1}^{r} M_{S_i}$ be its isotypic decomposition. Prove that any simple A-module is isomorphic to one S_i.

2.2 The Wedderburn classifications theorem

Example 2.2.1. Let D be a finite dimensional division $\mathbb{K}$-algebra. Then $D^{\oplus n} = \left\{ \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \mid d_1, \ldots, d_n \in D \right\}$ is a left $M_n(D)$-module. Then ${}_{M_n(D)}D^{\oplus n}$ is simple because if $s \in D^{\oplus n}$, $s \neq 0$, then $s_i \neq 0$ for some i, then $\begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & s_i^{-1} & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} s =$

$\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}$, and $\begin{pmatrix}0 & \cdots & t_1 & \cdots & 0\\ \vdots & & \vdots & & \vdots\\ 0 & \cdots & t_n & \cdots & 0\end{pmatrix}\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix} = \begin{pmatrix}t_1\\ \vdots\\ t_n\end{pmatrix}$, and so $M_n(D)s = D^{\oplus n}$. Now, $M_n(D)$ is a simple $\mathbb{K}$-algebra because

$$M_n(D) = \underbrace{\begin{pmatrix}* & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ * & 0 & \cdots & 0\end{pmatrix}}_{\cong D^{\oplus n}} \oplus \underbrace{\begin{pmatrix}0 & * & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & * & \cdots & 0\end{pmatrix}}_{\cong D^{\oplus n}} \oplus \cdots \oplus \underbrace{\begin{pmatrix}0 & \cdots & 0 & *\\ \vdots & & \vdots & \vdots\\ 0 & \cdots & 0 & *\end{pmatrix}}_{\cong D^{\oplus n}}.$$

Exercise 2.2.2. Any product $\prod_i M_{n_i}(D_i)$ is semisimple, where D_i is a division $\mathbb{K}$-algebra.

Lemma 2.2.3. *Let P and Q be two left A-modules, and suppose that $P = \bigoplus_{i=1}^{p} X_i$, $Q = \bigoplus_{i=1}^{q} Y_i$. Let $\varepsilon_j : X_j \hookrightarrow P$ be the inclusions, and $\pi_i : Q \twoheadrightarrow Y_i$ be the projections. Denote* $\mathrm{Hom} = \mathrm{Hom}_A$.

1. *Define*

$$M = \begin{pmatrix}\mathrm{Hom}(X_1,Y_1) & \cdots & \mathrm{Hom}(X_p,Y_1)\\ \vdots & \ddots & \vdots\\ \mathrm{Hom}(X_1,Y_q) & \cdots & \mathrm{Hom}(X_p,Y_q)\end{pmatrix} \cong \bigoplus_{i,j}\mathrm{Hom}(X_i,Y_j).$$

   Then $M \cong \mathrm{Hom}(P,Q)$ as A-modules.

2. *If $P = Q$, $p = q$, $X_i = Y_i$, then the isomorphism of the previous point is an isomorphism of $\mathbb{K}$-algebras :* $\mathrm{End}\,P \cong M$ *as rings, where M is endowed with the usual matrix multiplication.*

Proof. 1. Remark that

$$\begin{aligned}\phi : \mathrm{Hom}(P,Q) &\longrightarrow M\\ f &\longmapsto \begin{pmatrix}\pi_1 f\varepsilon_1 & \cdots & \pi_1 f\varepsilon_p\\ \vdots & \ddots & \vdots\\ \pi_q f\varepsilon_1 & \cdots & \pi_q f\varepsilon_p\end{pmatrix}\end{aligned}$$

is an isomorphism with inverse

$$\begin{aligned}
\psi : M &\longrightarrow \mathrm{Hom}(P,Q) \\
\begin{pmatrix} b_{1,1} & \cdots & b_{p,1} \\ \vdots & \ddots & \vdots \\ b_{1,q} & \cdots & b_{p,q} \end{pmatrix} &\longmapsto \psi((b_{i,j})_{i,j}), \\
\psi((b_{i,j})_{i,j}) : P &\longrightarrow Q \\
(x_1,\dots,x_p) &\longmapsto \left(\sum_{j=1}^{q} b_{1,j}(x_j),\dots,\sum_{j=1}^{q} b_{p,j}(x_j)\right).
\end{aligned}$$

2. The fact that $P=Q$, $p=q$, $X_i=Y_i$ make the matrix multiplication of M well defined. The rest is routine verifications.

□

Corollary 2.2.4. *1. Let S and T be two nonisomorphic simple A-modules. Then* $\mathrm{Hom}_A(S^{\oplus p},T^{\oplus q})=0$.

2. Let S be an A-module (not necessarily simple). Then $\mathrm{End}_A(S^{\oplus p})\cong M_p(\mathrm{End}_A S)$.

Proof. 1. By previous lemma, we have $\mathrm{Hom}_A(S^{\oplus p},T^{\oplus q})\cong M_{q\times p}(\underbrace{\mathrm{Hom}_A(S,T)}_{=0})=0$.

2. By previous lemma.

□

Theorem 2.2.5 (Wedderburn)**.** *Let A be a finite dimensional* $\mathbb{K}$*-algebra. If A is semisimple, then* $A\cong\prod_{i=1}^r M_{n_i}(D_i)$*, where* D_i *is a division* $\mathbb{K}$*-algebra,* $n_i\geq 1$*. Moreover,* $D_i^{\mathsf{op}}\cong \mathrm{End}_A S_i$*, where* S_i *is a simple A-module.*

Proof. We have that ${}_AA$ is semisimple, and let $A\cong\bigoplus_{i=1}^r S_i^{\oplus n_i}$ be its isotypic decomposition. By the lemma,

$$\begin{aligned}
\mathrm{End}_A A &\cong \begin{pmatrix} \mathrm{End}_A S_1^{\oplus n_1} & & 0 \\ & \ddots & \\ 0 & & \mathrm{End}_A S_r^{\oplus n_r} \end{pmatrix} \\
&\cong \prod_{i=1}^r \mathrm{End}_A S_i^{\oplus n_i} \\
&\cong \prod_{i=1}^r M_{n_i}(\mathrm{End}_A S_i).
\end{aligned}$$

By Schur's lemma, $\mathrm{End}_A S_i$ is a division algebra. But we have an isomorphism of algebras $A^{\mathrm{op}} \xrightarrow{\cong} \mathrm{End}_A A$. Indeed :

$$\begin{aligned} A^{\mathrm{op}} &\longrightarrow \mathrm{End}_A A \\ b &\longmapsto m_b, \qquad\qquad \text{right multiplication by } b, \\ \mathrm{End}_A A &\longrightarrow A^{\mathrm{op}} \\ f &\longmapsto f(1), \end{aligned}$$

are mutually inverse. Hence,

$$A \cong (\mathrm{End}_A A)^{\mathrm{op}} \cong (\prod_{i=1}^{r} M_{n_i}(\mathrm{End}_A S_i))^{\mathrm{op}} = \prod_{i=1}^{r} M_{n_i} \underbrace{(\mathrm{End}_A S_i)^{\mathrm{op}}}_{=D_i}.$$

The isomorphism $M_n(R)^{\mathrm{op}} \longrightarrow M_n(R^{\mathrm{op}})$ is given by transposition. □

Corollary 2.2.6. *Suppose that $\mathbb{K}$ is algebraically closed. If A is a semisimple $\mathbb{K}$-algebra, then $A \cong \prod_{i=1}^{r} M_{n_i}(\mathbb{K})$.*

Proof. In this case, $\mathrm{End}_A S_i \cong \mathbb{K}$ by Schur's lemma, and so $D_i = M_{n_i}(\mathbb{K})^{\mathrm{op}} = M_{n_i}(\mathbb{K})$ because $M_{n_i}(\mathbb{K})$ is commutative. □

Corollary 2.2.7. *If A is a simple $\mathbb{K}$-algebra, then $A \cong M_n(D)$, where $D = (\mathrm{End}_A S)^{\mathrm{op}}$, where S is the unique simple A-module up to isomorphism.*

Theorem 2.2.8 (Maschke)**.** *Take G a finite group, and $\mathbb{K}$ a field. Then $\mathbb{K}G$ is semisimple if and only if $\mathrm{char} K \nmid |G|$, i.e. $|G| \neq 0$ in $\mathbb{K}$.*

Proof. $\Longrightarrow$ Suppose that $\mathbb{K}G$ is semisimple. Consider

$$\begin{aligned} \varepsilon : \mathbb{K}G &\longrightarrow \mathbb{K} \\ \sum_{g\in G} \lambda_g g &\longmapsto \sum_{g\in G} \lambda_g. \end{aligned}$$

This makes $\mathbb{K}$ into a $\mathbb{K}G$ module (with trivial G-action), and ε is $\mathbb{K}G$-linear surjective. By semisimplicity, ε splits (as $\mathbb{K}$ is projective). Let $\sigma : \mathbb{K} \longrightarrow \mathbb{K}G$ a $\mathbb{K}G$-linear section of ε. Put $\sigma(1) = \sum_{g\in G} \lambda_g g$. Take $h \in G$, then $h\sigma(1) = \sigma(h1) = \sigma(1)$, and so

$$\sum_{g\in G} \lambda_g hg = \sum_{g\in G} \lambda_g g, \qquad \forall h \in G.$$

Hence all λ_gs are equal. Write $\sigma(1) = \lambda \sum_{g\in G} g$. Then

$$1 = \varepsilon\sigma(1) = \lambda\varepsilon \left(\sum_{g\in G} g \right) = \lambda|G|,$$

and so $|G| \neq 0$ in $\mathbb{K}$.

$\Longleftarrow$ Suppose $\operatorname{char}\mathbb{K} \not| |G|$. We prove that every $\mathbb{K}G$-module P is projective. Let $0 \to N \overset{\alpha}{\to} M \overset{\pi}{\twoheadrightarrow} P \to 0$ be a short exact sequence. The sequence splits as a sequence of $\mathbb{K}$-vector spaces and so there exists $\sigma : P \longrightarrow M$ a $\mathbb{K}$-linear section of π. Define $\tau = \frac{1}{|G|}\sum_{g\in G} g\sigma(g^{-1}\cdot -) : P \longrightarrow M$. It is easy to see that τ is a $\mathbb{K}G$-linear section of π, so the sequence splits, and P is projective. □

Exercise 2.2.9. Let A be a finite dimentional semisimple $\mathbb{K}$-algebra, and S_i be a simple A-module. Prove that the multiplicity n_i of S_i in the semisimple decomposition of A is $\dim_{D_i} S_i$, where $D_i = \operatorname{End}_A S_i$.

Chapter 3

The Jacobson radical

3.1 Definition

Definition 3.1.1 (Composition series)**.** Let M be a an R-module, where R is a ring. A *composition serie* of M is a sequence of submodules

$$0 = M_k < M_{k-1} < \cdots < M_0 = M$$

such that every quotient M_i/M_{i-1} is simple. Such a quotient is called a *composition factor*.

Lemma 3.1.2. *If A is a finite dimentional $\mathbb{K}$-algebra, and M a finitely generated A-module, then it admits a composition serie.*

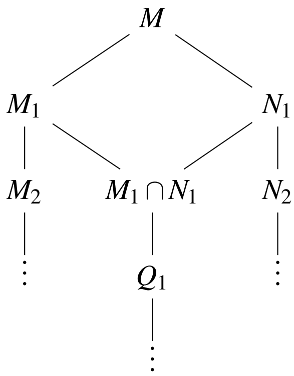
Proof. Recall that M is finite dimensional. Take a proper submodule N of maximal dimension. Then M/N is simple. Repeat with N. The process enventually stops as M is finite dimensional. □

Theorem 3.1.3 (Jordan–Holder)**.** *Let A be a finite dimensional $\mathbb{K}$-algebra, and M a finitely generated left A-module. Any two composition series of M have the same length and isomorphic quotients up to permutation. Explicitely, if*

$$0 = M_r < M_{r-1} < \cdots < M_0 = M, \qquad 0 = N_s < N_{s-1} < \cdots < N_0 = M$$

are two composition series, then $r = s$, and there exists $\sigma \in \mathfrak{S}_r$ such that $N_{i-1}/N_i \cong M_{\sigma(i)-1}/M_{\sigma(i)}$.

Proof. By induction on $\dim_{\mathbb{K}} M$. If $\dim_{\mathbb{K}} M = 1$, then M is simple, and the result is obvious. If $M_1 = N_1$, then the result follows by induction hypothesis. Suppose now that $M_1 \neq N_1$. Then $M_1 + N_1 = M$ by maximality of M_1 and N_1.



Let $0 = Q_k < \cdots < Q_0 = M_1 \cap N_1$ be a composition series. The two composition series of M_1 give (by induction) $r-1 = k+1$, and isomorphic quotients. Therefore, the composition factor of M_1 are, up to isomorphism, $\{$composition factors of $M_1 \cap N_1\} \cup \{M_1/M_1 \cap N_1\}$. Similarily for N_1 : its composition factors are, up to isomorphis, $\{$composition factors of $M_1 \cap N_1\} \cup \{N_1/M_1 \cap N_1\}$. Then the factors of the first series for M are, up to isomorphism $\{$composition factors of $M_1 \cap N_1\} \cup \{M/M_1, M_1/M_1 \cap N_1\}$. For the second series, we have $\{$composition factors of $M_1 \cap N_1\} \cup \{M/N_1, N_1/M_1 \cap N_1\}$. Moreover, $M/M_1 \cong N_1/M_1 \cap N_1$, and $M/N_1 \cong M_1/M_1 \cap N_1$. So the two composition series have the same factors. □

Corollary 3.1.4. *There are finitely many simple A-modules, up to isomorphism.*

Proof. It S is simple, then $S \cong A/J$, for some maximal ideal J. Then S is a composition factor of the following composition series of $0 < \cdots < J < A$. However, A only has finitely many composition factors. □

Definition 3.1.5 (Jacobson radical)**.** Let A be a finite dimensional $\mathbb{K}$-algebra, and M a finitely generated A-module.

1. The *Jacobson radical* of M, written $J(M)$, is the intersection of all maximal submodules of M.
2. The *Jacobson radical* of A is $J(A) = J({}_AA)$, the intersection of all maximal left ideals.

Example 3.1.6. If M is semisimple, then $J(M) = 0$. Indeed, $M = \bigoplus_i S_i$, and so $\bigoplus_{i \neq j} S_i$ is a maximal submodule. Hence $J(M) \leq \bigcap_j \bigoplus_{i \neq j} S_i = 0$.

3.2 Caracterisations

Propositions 3.2.1. *1. Let I be a left ideal of A. Then I is maximal if and only if there exists S a simple module, and* $s \in S$, $s \neq 0$ *such that* $I = \mathrm{Ann}(s)$, *where* $\mathrm{Ann}(s) = \{a \in A \mid as = 0\}$ *is the* annihilator *of s.*

2. *$J(A) = \bigcap_{S\ simple} \operatorname{Ann}(S)$, where $\operatorname{Ann}(S) = \{a \in A \mid aS = 0\} = \bigcap_{s \in S} \operatorname{Ann}(s)$ is the* annihilator *of S.*

3. *$J(A)$ is a two sided ideal.*

Proof. 1. A/I is simple, and so $I = \operatorname{Ann}(1)$, for $1 \in A/I$. Conversely, if S is simple, $s \neq 0$, then $As = S$, and so $A/\operatorname{Ann}(s) \cong S$ using the first isomorphism theorem for the map $\phi : A \longrightarrow S, a \longmapsto as$.

2. We have

$$J(A) = \bigcap_{I \text{ max. left ideal}} I = \bigcap_{S \text{ simple}} \underbrace{\bigcap_{0 \neq s \in S} \operatorname{Ann}(s)}_{=\operatorname{Ann}(S)}.$$

3. Let $a \in J(A)$, and $b \in A$. Let S be a simple module. Then $abS = 0$, as bS is either S or 0. Hence, $ab \in \operatorname{Ann}(S)$, for all simple module S, and so $ab \in J(A)$. □

Proposition 3.2.2. *Let I be a left ideal of A. Then $I \leq J(A)$ if and only if $1 + I \leq A^\times$*

Proof. $\Longrightarrow$ $a \in I \leq J(A)$, and so $a \in \mathfrak{m}$, for every maximal ideal $\mathfrak{m}$ of A, and so $1 + a \notin \mathfrak{m}$ (otherwise, $1 \in \mathfrak{m}$). However, an element is non left invertible if and only if it is contained in some (without loss of generality maximal) left ideal. So $1 + a$ is left invertible : $\exists u \in A$ such that $u(1 + a) = 1$. Now $u = 1 - ua$, and $-ua \in I$, as I is a left ideal. The same argument applied to ua shows that $-ua$ is left invertible : $\exists v \in A$ such that $v(1 - ua) = 1$. However $1 - ua = u$, and so $vu = 1$. Therefore $1 + a = vu(1 + a) = v$, and so $1 + a$ has right inverse u.

$\Longleftarrow$ Let $\mathfrak{m}$ be a maximal left ideal of A. If $a \notin \mathfrak{m}$, then $Aa + \mathfrak{m} = A$, because $\mathfrak{m}$ is maximal. So $ba + m = 1$, for some $b \in A$, $m \in \mathfrak{m}$. Hence $1 - ba = m \in \mathfrak{m}$, but $1 - ba \in A^\times$ as $-ba \in I$, which is absurd. Therefore $a \in \mathfrak{m}$, and this holds for every maximal left ideal $\mathfrak{m}$, so $a \in J(A)$.

□

中山のレンマ **3.2.3.** *Let A be a finite dimensional $\mathbb{K}$-algebra, and M a finitely generated left A-module. Then $J(A)M = M$, then $M = 0$.*

Proof. Let $m_1, \ldots, m_r$ be a minimal set of generators of M. So $M = \sum_i Am_i$. Since $J(A)M = M$, the element m_r can be written $m_r = \sum a_i m_i$, with $a_i \in J(A)$. Therefore $a_1 m_1 + \cdots + a_{r-1} m_{r-1} = \underbrace{(1 - a_r)}_{\text{invertible}} m_r$. So $M = \sum_{i<r} a_i m_i$, a contradiction with minimality of $\{m_i\}_{1 \leq i \leq r}$. □

中山のレンマ **3.2.4** (Alternate form)**.** *Let A be a finite dimensional $\mathbb{K}$-algebra, M a finitely generated left A-module, and $N \leq M$. If $J(A)M + N = M$, then $N = M$.*

Proof. See exercise 2 from sheet 6. □

Theorem 3.2.5. *Let A be a finite dimensional $\mathbb{K}$-algebra. Then $J(A)$ is the smallest two sided ideal with semisimple quotient. Explicitely :*

1. *$A/J(A)$ is semisimple,*
2. *and if A/I is also semsimple, with I a two sided ideal, then $J(A) \leq I$.*

Proof. We claim that $J(A)$ is an intersection of finitely many maximal left ideals. Let $I = \bigcap_i \mathfrak{m}_i$ be a finite intersection of maximal left ideals, with $\operatorname{codim}_{\mathbb{K}} I = \dim_{\mathbb{K}} A - \dim_{\mathbb{K}} I$ as large as possible (bounded by $\dim_{\mathbb{K}} A$). If $\mathfrak{m}$ is a maximal left ideal, then $I \cap \mathfrak{m}$ is a finite intersection with larger codimension, and so by maximality of $\operatorname{codim}_{\mathbb{K}} I$ and $\mathfrak{m}$, we have $I \cap \mathfrak{m} = I$, hence $I \leq \mathfrak{m}$. This argument applies for every left ideal $\mathfrak{m}$, and so $J(A) \leq I \leq J(A)$, which proves the claim.

1. Now, $J(A) = \bigcap_i \mathfrak{m}_i$. So the obvious homomorphism $A/J(A) \longrightarrow \bigoplus_i A/\mathfrak{m}_i$ is injective. Each summand of the latter is simple, so the sum is semisimple. Therefore, $A/J(A)$ is isomorphic to a submodule of a semisimple module, so it is itself semisimple.
2. Let I be a two sided ideal such that $A/I = \bigoplus_i S_i$ is semisimple. Note that $\operatorname{Ann}(A/I) = I$, and so $I \leq \operatorname{Ann}(S_i)$. Hence, $J(A) \leq I$.

□

Corollary 3.2.6. *A is semisimple if and only of $J(A) = 0$.*

Example 3.2.7. $\mathbb{Z}$ is not semisimple, but $J(\mathbb{Z}) = \bigcap_{p \text{ prime}} \mathbb{Z}p = 0$.

Theorem 3.2.8. *Let A be a finite dimensional $\mathbb{K}$-algebra. Then $J(A)$ is the largest two sided nilpotent ideal. Explicitely*

1. *$J(A)$ is nilpotent,*
2. *if I is a two sided nilpotent ideal, then $I \leq J(A)$.*

Proof. 1. Consider a composition series $0 = M_r < \cdots < M_0 = A$. Then every M_i is a left ideal. Since M_i/M_{i+1} is simple, then $J(A)M_i/M_{i+1} = 0$, hence $J(A)M_i \leq M_{i+1}$. Therefore

$$\underbrace{J(A)^r A}_{=J(A)^r} = J(A)^r M_0 \leq J(A)^{r-1} M_1 \leq \cdots \leq M_r = 0.$$

2. Let S be a simple A-module. Then $IS \leq S$. If $IS = S$, then $I^n S = S$, and for n large enough, $I^n = 0$, so $S = 0$, a contradiction. So $IS = 0$. So $I \leq \bigcap_{S \text{ simple}} \mathrm{Ann}(S) = J(A)$.

□

Example 3.2.9. Take $A = \mathbb{K}[t]/(t^n)$, then $J(A) = (t)$. It is obviously nilpotent, and $A/J(A) \cong \mathbb{K}$ is semisimple.

Proposition 3.2.10. *Let M be a finitely generated A-module, where A is a finite dimensional $\mathbb{K}$-algebra.*

1. *If $N \leq M$ such that M/N is semisimple, then $J(M) \leq N$.*

2. *$J(M) = J(A)M$.*

3. *$M/J(M)$ is semisimple.*

Proof. 1. If M/N is semisimple, then $M/N = \bigoplus_i S_i$, and so $J(M/N) = 0$ (see example 3.1.6). So $\bigcap_{N \leq L \text{ max. submod.}} L = N$. So $J(M) \leq N$.

2. If $L \leq M$ is a maximal submodule, then $J(A)M/L = 0$, as M/L is simple. So $J(A)M \leq L$. Therefore $J(A)M \leq \bigcap_{L \text{ max. submod.}} L = J(M)$. Conversely, $M/(J(A)M)$ is a $A/J(A)$-module. The latter algebra is semisimple, and so is $M/(J(A)M) = \bigoplus_i S_i$, where S_i is a simple $A/J(A)$-module, hence a simple A-module [1]. So $M/(J(A)M)$ is a semisimple A-module. By previous point, $J(M) \leq J(A)M$.

3. Already proved.

□

3.3 Local rings

Definition 3.3.1 (Local ring)**.** Let R be a ring. Then R is *local* if R has a unique maximal left ideal.

Lemma 3.3.2. *Let R be a local ring, and let J be its unique maximal left ideal. Then :*

1. *J is two sided.*

2. *$J = R \setminus R^\times$.*

3. *R/J is a division ring.*

[1] That trick wouldn't work for an arbitrary restriction of scalars

Proof. 1. If $b \in R \setminus \{0\}$, then $\mathrm{Ann}(b)$ is a proper left ideal, as it doesn't contain 1, so $\mathrm{Ann}(b) \leq J$. If $Jb \not\leq J$, then we would get $Jb = R$. So b is left invertible, by an element $a \in J$. Then $(1-ba)b = b - bab = 0$. So $1 - ba \in \mathrm{Ann}(b) \leq J$. But $a \in J$, so $1 = \underbrace{1-ba}_{\in J} + \underbrace{ba}_{\in J} \in J$, which is absurd.

2. If $r \in R \setminus R^\times$, then r is contained in a proper ideal, namely (r). So it is contained is a maximal ideal (using Zorn's lemma), which nessecesarily is J. So $r \in J$, and $R \setminus R^\times \subseteq J$. Conversely, J cannot have any left unit (as it is a left ideal), nor right unit (as it is a right ideal), hence the equality.

3. Obvious.

□

Lemma 3.3.3. *Let R be a ring. Suppose that every element of R is either invertible or nilpotent. Then R is local.*

Proof. Recall that an invertible element cannot be nilpotent. Take J the set of all nilpotent elements of R. Let $a \in J$ and $b \in R$. The ba is a zero divisor : $ba \cdot a^{n-1} = ba^n = 0$, where n is the least integer such that $a^n = 0$. So ba is not invertible, hence nilpotent, hence $ba \in J$. Let $a_1, a_2 \in J$. Suppose $a_1 + a_2 \notin J$, hence invertible. So $xa_1 + xa_2 = 1$, for some x. Note that xa_1 and xa_2 commute (as r and $1-r$ always commute, for any $r \in R$). Let n_1 and n_2 be integers such that $(xa_i)^{n_i} = 0$, and $N = n_1 + n_2$. We can use Newton's formula (as xa_1 and xa_2 commute) :

$$1 = (xa_1 + xa_2)^N = \sum_{i=0}^{N} \binom{N}{i} \underbrace{(xa_1)^i (xa_2)^{N-i}}_{=0},$$

indeed, we ecessarily have that $i \geq n_1$ or $N - i \geq n_2$. We have an absurdity, so $a_1 + a_2 \in J$. Hence, J is an ideal, and $J = R \setminus R^\times$. So J is the unique maximal ideal, and R is local. □

Exercise 3.3.4. Let A be a finite dimentional $\mathbb{K}$-algebra. Then A is a local if and only if $A/J(A)$ is a division algebra.

Chapter 4

Indecomposable modules

4.1 The Krull–Remak–Schmidt decomposition theorem

Definition 4.1.1 ((In)decomposable module)**.** Let A be a finite dimentional $\mathbb{K}$-algebra, where $\mathbb{K}$ is a field. A left A-module M is *decomposable* if $M = M_1 \oplus M_2$, for $M_1, M_2 \leq M$ non zero. It is *indecomposable* otherwise, if it is not null.

Example 4.1.2. Any simple module is indecomposable.

Exercise 4.1.3. If conversely every indecomposable A-module is simple, then A is a semisimple algebra.

Remark 4.1.4. Every finitely generated A-module can be decomposed as a direct sum $M = \bigoplus_i M_i$ of indecomposable submodules.

Lemma 4.1.5 (Fitting's lemma)**.** *Let A be a finite dimentional $\mathbb{K}$-algebra, and M a finitely generated A-module. Let $\phi \in \mathrm{End}_A M$. Then there exists $n \in \mathbb{N}$ such that $M = \ker \phi^n \oplus \operatorname{im} \phi^n$.*

Proof. Since $\dim_{\mathbb{K}} M$ is finite, the following two series must stop :

$$M \geq \operatorname{im}\phi \geq \operatorname{im}\phi^2 \geq \cdots \geq \operatorname{im}\phi^n = \operatorname{im}\phi^{n+1} = \cdots,$$

$$0 \leq \ker\phi \leq \ker\phi^2 \leq \cdots \leq \ker\phi^n = \ker\phi^{n+1} = \cdots,$$

and $\operatorname{im}\phi^{n+k} = \operatorname{im}\phi^n$, $\ker\phi^{n+k} = \ker\phi^n$ for all $k \in \mathbb{N}$. Let $x \in M$. Then $\phi^n(x) = \phi^{2n}(y)$ for some $y \in M$ (as $\operatorname{im}\phi^n = \operatorname{im}\phi^{2n}$). So $\phi^n(x - \phi^n(y)) = 0$, therefore $x = \underbrace{\phi^n(y)}_{\in \operatorname{im}\phi^n} + \underbrace{x - \phi^n(y)}_{\in \ker\phi^n}$. So $M = \operatorname{im}\phi^n + \ker\phi^n$. Let $x \in \operatorname{im}\phi^n \cap \ker\phi^n$. Then $x = \phi^n(z)$ for some $z \in M$. Therefore $0 = \phi^n(x) = \phi^{2n}(z)$. So $z \in \ker\phi^{2n} = \ker\phi^n$, hence $x = 0$. Hence $\operatorname{im}\phi^n \cap \ker\phi^n = 0$, and we get $M = \ker\phi^n \oplus \operatorname{im}\phi^n$. ☐

Theorem 4.1.6. *Let A be a finite dimentional $\mathbb{K}$-algebra, and M a finitely generated left A-module. Then M is indecomposable if and only if $\mathrm{End}_A M$ is local.*

Proof. $\Longrightarrow$ Suppose M indecomposable. By Fitting's lemma, there exists an integer such that $M = \operatorname{im}\phi^n \oplus \ker\phi^n$, for a given $\phi \in \operatorname{End}_A M$. However, M is indecomposable, si one of the summands is zero. If $\ker\phi^n = 0$ and $\operatorname{im}\phi^n = M$, then $\ker\phi = 0$, and $\operatorname{im}\phi = M$, and so ϕ is an automorphism. If $\ker\phi^n = M$, then ϕ is nilpotent. By lemma 3.3.3, we obtain that $\operatorname{End}_A M$ is local.

$\Longleftarrow$ Suppose $\operatorname{End}_A M$ local. Let $M = M_1 \oplus M_2$, and $\pi_i : M \twoheadrightarrow M_i \hookrightarrow M$ be the projections. Then $\pi_i^2 = \pi_i$, $\pi_1 + \pi_2 = \mathrm{id}_M$. Without loss of generality, suppose $M_1 \neq 0$, then $\pi_1 \neq 0$, and so it is not nilpotent (because it is idempotent). Hence, π_1 is invertible. Then

$$\pi_1 = \pi_1^{-1} \underbrace{\pi_1^2}_{=\pi_1} = \mathrm{id}_M\,.$$

Hence $\pi_2 = 0$, and so $M_2 = 0$.

□

Corollary 4.1.7. *A is local if and only if ${}_AA$ is indecomposable.*

Proof. We have $\operatorname{End}_A({}_AA) \cong A^{\mathrm{op}}$ which is also local. □

Theorem 4.1.8 (Krull–Remak–Schmidt)**.** *Let A be a finitely generated $\mathbb{K}$-algebra, and M a finite dimensional left A-module. Suppose that $M = \bigoplus_{i=1}^r M_i = \bigoplus_{j=1}^s N_j$, where all summands are indecomposable. Then $r = s$, and there exists $\sigma \in \mathfrak{S}_r$ such that $M_i \cong N_{\sigma(i)}$. In other words, a decomposition into indecomposable modules is essentially unique.*

Proof. We proceed by induction on r. If $r = 1$, then $M = M_1$ is indecomposable, so $s = 1$, and $M_1 = N_1$. Suppose $r \geq 2$. Let $\varepsilon_i : M_i \hookrightarrow M$ the canonical inclusion, $\pi_i : M \twoheadrightarrow M_i$ the canonical projection, and $\eta_j : N_j \hookrightarrow M$, $\rho_j : M \twoheadrightarrow N_j$ the analogous. Then $\pi_i\varepsilon_i = \mathrm{id}_{M_i}$, and $\varepsilon_i\pi_i : M \longrightarrow M$ is idempotent. Moreover $\sum_i \varepsilon_i\pi_i = \mathrm{id}_M$. Similarily for η_j and ρ_j.

The composite $M_1 \xhookrightarrow{\varepsilon_1} M \xrightarrow{\sum_j \eta_j\rho_j} M \xtwoheadrightarrow{\pi_1} M$ is id_{M_1}. Therefore $\mathrm{id}_{M_1} = \sum_j \pi_1\eta_j\rho_j\varepsilon_1 \in \operatorname{End}_A M_1$, and the latter ring is local. So it can't be that all summands are nilpotent. Let $1 \leq j \leq s$ such that $\phi = \pi_1\eta_j\rho_j\varepsilon_1$ is invertible. Without loss of generality (i.e. up to permutation), $j = 1$.

Let $\alpha = \rho_1\varepsilon_1\phi^{-1}$ and $\beta = \pi_1\eta_1$. The composite $M_1 \xrightarrow{\alpha} N_1 \xrightarrow{\beta} M_1$ is id_{M_1}. Moreover $\alpha\beta$ is idempotent and not zero (because $\beta(\alpha\beta)\alpha = \mathrm{id}_{M_1}$). It is therefore not nilpotent, so it is invertible, as $\operatorname{End}_A N_1$ is local. Denote $\gamma = \alpha\beta$. Then $\gamma = \gamma^{-1}\gamma^2 = \gamma^{-1}\gamma = \mathrm{id}_{N_1}$. So α and β are mutually inverse, hence $M_1 \cong N_1$.

Now, $\bigoplus_{i=2}^r M_i \cong \bigoplus_{j=2}^s N_i$, and we apply the induction hypothesis. □

Remarks 4.1.9. 1. The Krull–Remak–Schmidt theorem fails for some other rings, e.g. for ring of integers in a number field (a finite field extension of $\mathbb{Q}$), e.g. $\mathbb{Z}[\sqrt{5}]$.

2. But it works for PIDs (yayyyy).

3. If A is a finitely generated $\mathbb{K}$-algebra, and M is a finite dimensional left A-module, then we know that $\dim_{\mathbb{K}} M < \infty$. Therefore a decomposition into indecomposable always exists (by induction).

4.2 Idempotent elements

Definition 4.2.1 (Orthogonal/primitive idempotents)**.** Let R be a ring, and $e, f \in R$ two idempotents. They are called *orthogonal* if $ef = fe = 0$. An idempotent $e \in R$ is called *primitive* if e cannot be decomposed as a sum of two nonzero orthogonal idempotents.

Example 4.2.2. In any ring R, and for all idempotent $e \in R$, we have that e and $(1-e)$ are orthogonal idempotents. Hence, if $e \neq 0, 1$, $e + (1-e) = 1$ is an orthogonal decomposition, and 1 is not primitive.

Lemma 4.2.3. *Let R be a ring.*

1. *If* ${}_RR = \bigoplus_{i=1}^r Q_i$*, then* $Q_i = Re_i$*, where* e_i *is idempotent, and* $\sum_i e_i = 1$ *is an orthogonal decomposition.*

2. *Conversely, any orthogonal decomposition of* 1 *into idempotents* $e_1, \dots, e_r$ *leads to a decomposition* ${}_RR = \bigoplus_{i=1}^r Re_i$*.*

3. *If* $e \in R$ *is idempotent, then* Re *is indecomposable if and only if* e *is primitive.*

Proof.

1. There is a unique way of writing $1 = \sum_i e_i$, where $e_i \in Q_i$. Then $e_j = \sum_i \underbrace{e_j e_i}_{\in Q_i}$, and so $e_j e_i = 0$ if $i \neq j$, and $e_j^2 = e_j$. So $\{e_i\}_i$ is a family of orthogonal idempotents. Take $x \in Q_j$. Then $x = x1 = \sum_i xe_i$, and so $xe_i = 0$ if $i \neq j$, and $xe_j = x$. So $Q_j = Re_j$.

2. We have $1 = \sum_i e_i$, so $R = R1 = \sum_i Re_i$. Let $x \in Re_j \cap \sum_{i \neq j} Re_i$. Then $x = xe_i$, and $x = \sum_{i \neq j} j_i e_j$. So $x = xe_i = \sum_{i \neq j} j_i e_j e_i = 0$.

3. If $e = u + v$ is a decomposition into orthogonal idempotents, then $Re = Ru \oplus Rv$. Conversely, if $Re = U \oplus V$, then $e = \pi_U(e) + \pi_V(e)$, and by the same argument as in point 1., those two terms are orthogonal idempotents.

□

Corollary 4.2.4. *Let A be a finite dimentional* $\mathbb{K}$*-algebra. Let P be a finitely generated projective left A-module.*

1. *P is indecomposable if and only if* $P \cong Ae$*, where e is a primitive idempotent of A.*

2. *There are finitely many indecomposable projective modules up to isomorphism.*

Proof. First, decompose ${}_AA$. By the previous lemma, $A = \bigoplus_{i=1}^n Ae_i$, where $1 = \sum_i e_i$ is an orthogonal primitive idempotent decomposition. Such a decomposition must exist because of the Krull–Remak–Schmidt theorem.

1. P is a direct summand of $A^{\oplus r}$ which decomposes as $\bigoplus_{i=1}^n (Ae_i)^{\oplus r}$. By the Krull–Remak–Schmidt theorem, $P \cong Ae_i$ for some i.

2. We know that $A^{\oplus r} \cong \bigoplus_{i=1}^n (Ae_i)^{\oplus r}$, and so any indecomposable projective module is isomorphic to Ae_i, for some e_i.

□

Theorem 4.2.5. *Let A be a finite dimentional $\mathbb{K}$-algebra. There is a bijection*

$$\left\{\begin{matrix}\text{conjugacy classes of primitive}\\ \text{idempotents in } A\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{iso. classes of indecomposable}\\ \text{projective left } A\text{-modules}\end{matrix}\right\}$$

mapping the class of a primitive idempotent $e \in A$ to the isomorphism class of Ae.

Proof. We have to check that this mapping is well defined. It is clear that any conjugate of an idempotent (resp. primitive idempotent) is again idempotent (resp. primitive idempotent). Let $e, f \in A$ be conjugate primitive idempotents. There exists $u \in A^\times$ such that $e = ufu^{-1}$. Define $m_u : Af \longrightarrow Ae : x \longmapsto xu$, and $m_{u^{-1}} : Ae \longrightarrow Af$ similarily. Then m_u and $m_{u^{-1}}$ are mutually inverse, and so $Ae \cong Af$.

Then, the mapping is surjective by the previous corollary. It remains to show injectivity. Let $e, f \in A$ be primitive idempotents such that $Ae \cong Af$. We have $A \cong Ae \oplus A(1-e) \cong Af \oplus A(1-f)$. So $A(1-e) \cong A(1-f)$ by cancellation (exercise 2, sheet 8). Therefore there exists an isomorphism $\phi : A \longrightarrow A$ such that $\phi(Ae) = Af$, and $\phi(A(1-e)) = A(1-f)$. However, ϕ must be of the form $\phi = m_u$ (right multiplication by u), for some $u \in A^\times$. Then $u^{-1}eu$ is an idempotents, and $Au^{-1}eu = Aeu = \phi(Ae) = Af$, and $A(1-u^{-1}eu) = Au^{-1}(1-e)u = \phi(A(1-e)) = A(1-f)$. Then m_f and $m_{u^{-1}eu}$ are the identity on Af, and zero on $A(1-f)$. Therefore $m_f = m_{u^{-1}eu}$ on $A = Af \oplus A(1-f)$. In particular, they coincide on 1, and so $f = u^{-1}eu$. □

Example 4.2.6. 1. Take A to be semisimple. By the Wedderburn theorem, $A \cong \prod_i M_{n_i}(D_i)$, where D_i is a division ring. Take

$$e_i = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_{n_i}(D_i).$$

It is a primitive idempotent. Moreover, as A is semisimple, any indecomposable projective is simple. Hence

$$Ae_i \cong M_{n_i}(D_i)e_i = \begin{pmatrix} D_i \\ \vdots \\ D_i \end{pmatrix} \cong D_i^n$$

is simple. See exercises.

2. An example with infinitely many indecomposable modules. Define the algebra $A = \mathbb{K}[X,Y]/(X^2,Y^2,XY)$. Let $s = \bar{X}$, $t = \bar{Y}$. Then $\dim_{\mathbb{K}} A = 3$, with basis $\{1,s,t\}$, and multiplication $s^2 = t^2 = st = ts = 0$. Clearly, $J(A) = \mathbb{K}s \oplus \mathbb{K}t$, as it is the largest nilpotent ideal. Then, $A/J(A) = \mathbb{K}$. We have only one simple module $S = A/J(A)$, one idempotent, namely 1, which is also primitive. Therefore, A is indecomposable projective. Fix $\lambda \in \mathbb{K}^\times$, and define a 2-dimentional A-module M_λ with basis $\{m,n\}$ as follows :

$$\begin{aligned} sm = n, &\quad sn = 0 \\ tm = \lambda n, &\quad tn = 0. \end{aligned}$$

Let $N_\lambda = J(M_\lambda) = J(A)M_\lambda = \mathbb{K}n$. We have $N_\lambda \cong S$, and $M_\lambda/N_\lambda \cong S$, and so there is a short exact sequence

$$0 \longrightarrow S \longrightarrow M_\lambda \longrightarrow S \longrightarrow 0.$$

The rest of the proof that M_λ is indecomposable is left as an exercise. Next, $m_s : M_\lambda \longrightarrow M_\lambda : x \longmapsto sx$ induces a linear map $\bar{m}_s : M_\lambda/N_\lambda \longrightarrow N_\lambda : s\bar{x} \longmapsto x$. Moreover $\bar{m}_s(m) = n$, and so $\bar{m}_s$ is an isomorphism. Similariy, $m_t : M_\lambda \longrightarrow M_\lambda$ induces a $\mathbb{K}$-linear isomorphism $\bar{m}_t : M_\lambda/N_\lambda \longrightarrow N_\lambda$, with $\bar{m}_t(m) = \lambda n$. Now the composite ϕ_λ

$$N_\lambda = J(M_\lambda) \xrightarrow{\bar{m}_t^{-1}} M_\lambda/J(M_\lambda) \xrightarrow{\bar{m}_s} N_\lambda$$

maps n to $\lambda^{-1}n$. The map ϕ_λ is intrisically defined by M_λ, because it is $\bar{m}_s\bar{m}_t^{-1}$. Assiciated to M_λ we have a intrisically defined map $\phi_\lambda : N_\lambda = \mathbb{K}n \longrightarrow N_\lambda$ which is multiplication by λ^{-1}. So from M_λ, one can recover λ. Therefore if $M_\lambda \cong M_\mu$, then $\lambda = \mu$. Finally, if $\mathbb{K}$ is infinite, we have infinitely many non isomorphic indecomposable modules.

Chapter 5

Lifting idempotents

Definition 5.0.7 (Primitive orthogonal decomposition)**.** Let R be a ring. A *primitive orthogonal decomposition* of an idempotent $e \in R$ is a decomposition of e as a sum of primitive pairwise orthogonal idempotents.

Lemma 5.0.8. *Let A be a finit dimentional $\mathbb{K}$-algebra, $e \in A$ idempotent. Consider the $\mathbb{K}$-algebra eAe (with identity e). Then $J(eAe) = eJ(A)e$.*

Proof. First, $eJ(A)e \subseteq J(A), eAe$. Next, if $a \in J(A) \cap eAe$, then $a = ebe$ with $b \in A$, and so $a = eae$. Therefore $a \in eJ(A)e$. Hence, $eJ(A)e = J(A) \cap eAe$.

Moreover, $J(A)^N = 0$ for some lage enough $N \in \mathbb{N}$, so $(eJ(A)e)^N = 0$. So the two sided ideal $eJ(A)e$ of eAe is nilpotent, hence $eJ(A)e \subseteq J(eAe)$. Next, $AJ(eAe)A$ is the two sided ideal of A denerated by $J(eAe)$. We have

$$(AJ(eAe)A)^2 = Ae\underbrace{J(eAe)eAeJ(eAe)}_{=J(eAe)^2}eA.$$

Similarily, $(AJ(eAe)A)^n = AeJ(eAe)^n eA$, hence, $AJ(eAe)A$ is nilpotent, and so $AJ(eAe)A \subseteq J(A)$. Therefore, $J(eAe) \subseteq J(A)$, hence $J(eAe) \subseteq eJ(A)e$.

Finally, $J(eAe) = eJ(A)e$. □

Theorem 5.0.9 (Lifting stuffs)**.** *Let A be a finite dimentional $\mathbb{K}$-algebra, $\bar{A} = A/J(A)$, and write $\bar{a} \in \bar{A}$ for the class of $a \in A$ in $\bar{A}$.*

1. ***Lifting invertibility** : $a \in A$ is invertible if and only if $\bar{a}$ is invertible. In other words, there is a short exact sequence of groups*

$$\{1\} \longrightarrow 1 + J(A) \longrightarrow A^\times \longrightarrow \bar{A}^\times \longrightarrow \{1\}.$$

2. ***Lifting idempotents** : For any idempotent $g \in \bar{A}$, there exists an idempotent $e \in A$ such that $\bar{e} = g$.*

3. ***Lifting conjugacy of idempotents** : Let $e, f \in A$ be two idempotents, if $\bar{e}$ and $\bar{f}$ are conjugate in $\bar{A}$, then e and f are conjugate in A. More precisely, if $\bar{f} = \bar{u}\bar{e}\bar{u}^{-1}$, then $\bar{u}$ can be lifted as $u \in A^\times$ such that $f = ueu^{-1}$. In particular, if $\bar{e} = \bar{f}$, then there exists $u \in 1 + J(A)$ such that $f = ueu^{-1}$.*

4. ***Lifting primitivity*** *: Let $e \in A$ be idempotent. Then e is primitive in A if and only if $\bar{e}$ is primitive in $\bar{A}$.*

5. ***Lifting idempotent decompositions*** *: Let $e \in A$ be idempotent, and $\bar{e} = \sum_i g_i$ be a primitive orthogonal decomposition of $\bar{E}$ in $\bar{A}$. Then there exists $e_i \in A$ with $\bar{e}_i = g_i$, and such that $e = \sum_i e_i$ is a primitive orthogonal decomposition of e in A.*

Proof. 1. If $a \in A$ is invertible, then clearly $\bar{a}$ is also invertible. Conversely, take $a \in A$ such that $\bar{a}$ is invertible, and suppose that a is not invertible. If a is not in a maximal left ideal, then $Aa = A$, and similarily on the right. If $Aa = aA = A$, then $\exists b, c \in A$ such that $ba = ac = 1$, so a is invertible, a contradiction. Therefore, a belongs to a maximal left ideal $\mathfrak{m} \subseteq A$. By definition, $J(A) \subseteq \mathfrak{m}$. Hence $\overline{\mathfrak{m}} = \mathfrak{m}/J(A)$ is a maximal ideal in $\bar{A}$, and $\bar{a} \in \overline{\mathfrak{m}}$. Consequently $\bar{a}$ is not invertible, a contradiction.

Finally, the projection $A^\times \longrightarrow \bar{A}^\times$ is surjective, with kernel $1 + J(A)$.

2. Let $g \in \bar{A}$ be an idempotent. By surjectivity of the projection $A \twoheadrightarrow \bar{A}$, there exists $a_1 \in A$ such that $\bar{a}_1 = g$. Let $b_1 = a_1^2 - a_1$. Define inductively $a_n = a_{n-1} + b_{n-1} - 2a_{n-1}b_{n-1}$, and $b_n = a_n^2 - a_n$.

We show by induction that $b_n \in J(A)^n$. For $n = 1$, remark that $\bar{b}_1 = g^2 - g = 0$, and so $b_1 \in J(A)$. We use the fact that $a_n^2 = a_n + b_n$, and also that a_n and b_n commute (as $b_n = a_n^2 - a_n$). By induction, assume that $b_n \in J(A)^n$. Notice that $b_n^2 \in J(A)^{2n} \subseteq J(A)^{n+1}$. Compute a_{n+1} modulo $J(A)^{n+1}$.

$$\begin{aligned}
a_{n+1}^2 &= (a_n + b_n - 2a_nb_n)^2 \\
&= a_n^2 + b_n^2 + 4a_n^2b_n^2 + 2a_nb_n - 4a_n^2b_n - 4a_nb_n^2 \\
&\equiv a_n^2 + 2a_nb_n - 4a_n^2b_n && \mod J(A)^{n+1} \\
&\equiv a_n + b_n + 2a_nb_n - 4(a_n + b_n)b_n && \mod J(A)^{n+1} \\
&\equiv a_n + b_n - 2a_nb_n && \mod J(A)^{n+1} \\
&\equiv a_{n+1} && \mod J(A)^{n+1}.
\end{aligned}$$

Therefore, $a_{n+1}^2 \equiv a_{n+1} \mod J(A)^{n+1}$, and so $b_{n+1} = a_{n+1}^2 - a_{n+1} \in J(A)^{n+1}$, thus proving the claim.

As $J(A)$ is nilpotent, there exists, $N \in \mathbb{N}$ such that $J(A)^N = 0$, and so $b_N = a_N^2 - a_N = 0$, and a_N is idempotent. By induction, on can easily prove that $\bar{a}_N = g$, which proves the statement.

3. Let $e, f \in A$ be two idempotents such that $\bar{f} = \bar{u}\bar{e}\bar{u}^{-1}$ for some $\bar{u} \in \bar{a}^\times$. Then $\bar{u}$ lifts as $u \in A^\times$ by part 1. Denote by $h = ueu^{-1}$. It is an idempotent, and $\bar{h} = \bar{f}$. Let $v = 1 - h - f + 2hf$. Then $\bar{v} = 1 - \bar{h} - \bar{f} + 2\bar{f}\bar{h} = 1$. So $v \in 1 + J(A)$ is invertible. Compute

$$\begin{aligned}
hv &= h - h - hf + 2hf = hf \\
vf &= f - hf - f + 2hf = hf.
\end{aligned}$$

So $hv = vf$, and $ueu^{-1} = h = vfv^{-1}$, and e and f are conjugate.

4. Let $e \in A$ to be an idempotent. If e is not primitive, then $e = f_1 + f_2$, for $f_1, f_2 \in A$ orthogonal idempotents. Remark that $f_1, f_2 \notin J(A)$ as $J(A)$ is nilpotent. So $\bar{f}_1, \bar{f}_2 \neq 0$, and as $\bar{e} = \bar{f}_1 + \bar{f}_2$ is an orthogonal idempotent decomposition, we have that $\bar{e}$ is not primitive.

   Conversely, if $\bar{e}$ is not primitive, then $\bar{e} = \bar{f}_1 + \bar{f}_2$, for $\bar{f}_1, \bar{f}_2 \in \bar{a}$ orthognal idempotents. Consider the $\mathbb{K}$-algebra eAe. By previous lemma, we have $\overline{eAe} = \bar{e}\bar{A}\bar{e}$. Notice that $\bar{f}_1, \bar{f}_2 \in \bar{e}\bar{A}\bar{e}$. By part 2, $\bar{f}_1$ can be lifted as an idempotent $f_1 \in eAe$. Define $f_2 = e - f_1$, which lifts $\bar{f}_2$. Then $e = f_1 + f_2$ is an orthogonal decomposition of e, and it is not primitive.

5. Let $e \in A$ be an idempotent, and $\bar{e} = \sum_{i=1}^{r} g_i$ be an orthogonal decomposition. We prove the statement by induction on r. If $r = 1$ there is nothing to prove. We work in the $\mathbb{K}$-algebra eAe. We have that $g_1 \in \bar{e}\bar{a}\bar{e}$ as before, and we can lift it as an idempotent $f_1 \in eAe$. Then $e = f_1 + (e - f_1)$ is an orthogonal decomposition (in eAe). Hence $\bar{e} = g_1 + \underbrace{\sum_{i=2}^{r} g_r}_{=\overline{e-f_1}}$. By induction hypothesis, the decomposition $\overline{e - f_1} = \sum_{i=2}^{r} g_i$ lifts as an orthogonal decomposition $e - f_1 = \sum_{i=2}^{r} f_i$. So $e = \sum_{i=1}^{r} f_i$ is an orthogonal decomposition lifting $\bar{e} = \sum_{i=1}^{r} g_i$.

□

Theorem 5.0.10. *Let A be a finite dimentional $\mathbb{K}$-algebra.*

1. *If $e \in A$ is a primitive idempotent, then the indecomposable projective A-module Ae has a unique maximal submodule $J(A)e$. In other words, Ae has a unique simple quotient, namely $Ae/J(A)e = \bar{A}\bar{e}$.*

2. *Let $e, f \in A$ be two primitive idempotents, then $Ae \cong Af$ if and only if $\bar{A}\bar{e} \cong \bar{A}\bar{f}$.*

3. *There is a bijection*

$$\left\{ \begin{array}{c} \textit{iso. classes of} \\ \textit{indec. projective A-mods.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{iso. classes of} \\ \textit{simple A-mods.} \end{array} \right\}$$

   mapping the class of Ae to the class of $\bar{A}\bar{e}$.

Proof. The $\mathbb{K}$-algebra projection map $A \longrightarrow \bar{A}$ sends Ae to $\bar{A}\bar{e}$. If e is primitive, then so is $\bar{e}$, by previous theorem. So $\bar{A}\bar{e}$ is a projective indecomposable $\bar{A}$-module. However, $\bar{A}$ is semisimple. So $\bar{A}\bar{e}$ is simple as a $\bar{A}$-module, and also as a A-module (with $J(A)$ acting as zero, the classical restriction of scalars). Moreover, any simple A-module has this form, because $J(A)$ must act by zero on it (by definition of the Jacobson radical).

1. We have seen that $J(A)e$ is a maximal submodule of Ae, because $\bar{A}\bar{e}$ is simple. Let M be a maximal submodule of Ae. Then Ae/M is simple, therefore $J(A)\,(Ae/M) = 0$, and so $J(A)Ae \leq M$, that is $J(A)e \leq M \leq Ae$. Since $J(A)e$ is maximal, we have $M = J(A)e$.

2. We use the bijection of theorem 4.2.5. We have $Ae \cong Af$ if and only if e and f are conjugate in A, if and only if $\bar{e}$ and $\bar{f}$ are conjugate in $\bar{A}$, if and only if $\bar{A}\bar{e} \cong \bar{A}\bar{f}$.

3. This is a consequence of the previous points.

□

Chapter 6

The Wedderburn–Malcev theorem

Definition 6.0.11 (Split algebra)**.** Let A be a finite dimentional $\mathbb{K}$-algebra. It is said to be *split* if

1. it is semisimple,

2. $\operatorname{End}_A S = \mathbb{K}$, for every simple A-module S.

Remark 6.0.12. Take A a semisimple $\mathbb{K}$-algebra. The by Wedderburn theorem, we have that

$$A \cong \prod_i M_{n_i}(D_i).$$

Then A is split if and only if $D_i = \mathbb{K}$, since $D_i = (\operatorname{End}_A S_i)^{\mathrm{op}}$.

Examples 6.0.13.

1. If $\mathbb{K}$ is algebraically closed, then every semisimple algebra is split, by Schur's lemma.

2. Consider $\mathbb{R}Q_8$, the $\mathbb{R}$-group algebra of the quaternion group of order 8. Then $\mathbb{R}Q_8 \cong \mathbb{R}^4 \times \mathbb{H}$, which is non split (see exercise 2 of sheet 5).

Lemma 6.0.14. *Let $A = A_1 \times A_2$ be a direct product of two finite dimentional $\mathbb{K}$-algebras. Let $e \in A$ be a primitive idempotent.*

1. *Then either $e = (e_1, 0)$, where $e_1 \in A_1$ is a primitive idempotent, or $e = (0, e_2)$, where $e_2 \in A_2$ is a primitive idempotent.*

2. *A primitive orthogonal decomposition of $1 \in A$ is obtained by a primitive orthogonal decomposition of $1 \in A_1$ and $1 \in A_2$.*

Proof.

1. Let $e \in (e_1, e_2) \in A$ be a primitive idempotent. Then $e^2 = e$, so e_1 and e_2 are idempotents, and so are $(e_1, 0)$ and $(0, e_2)$. Moreover, $e = (e_1, 0) + (0, e_2)$ is an orthogonal decomposition, so either $e_1 = 0$, or $e_2 = 0$.

2. Clear since $1_A = (1_{A_1}, 1_{A_2}) = (1_{A_1}, 0) + (0, 1_{A_2})$ is an (not necessarily primitive) orthogonal decomposition.

□

Lemma 6.0.15. *Let $A = M_n(\mathbb{K})$ be a split simple algebra over $\mathbb{K}$.*

1. *An idempotent $e \in A$ is primitive if and only if e is a projection matrix onto a one dimentional subspace of $\mathbb{K}^n$.*
2. *Every primitive idempotents are conjugate.*
3. *The number of primitive idempotents in a primitive orthogonal decomposition of I_n is n. More precisely, $I_n = \sum_{j=1}^{n} E_{j,j}$. By part 2, there exists $U_j \in \mathrm{GL}_n(\mathbb{K})$ such that $E_{j,j} = U_j^{-1} E_{1,1} U_j$.*
4. *The elements $\{U_j^{-1} E_{1,1} U_k\}$ form a $\mathbb{K}$-basis of $M_n(\mathbb{K})$. If the U_j are transposition matrices, then the basis is precisely the canonical basis.*

Proof. Already done in an exercise sheet. For point 4. : if U_j are permutation matrices, then the result is clear. If not, then

$$\begin{aligned}(U_j^{-1} E_{1,1} U_k)(U_p^{-1} E_{1,1} U_q) &= U_j^{-1} U_k (U_k^{-1} E_{1,1} U_k)(U_p^{-1} E_{1,1} U_p) U_p^{-1} U_q \\ &= U_j^{-1} U_k E_{k,k} E_{p,p} U_p^{-1} U_q\end{aligned}$$

is either 0 os another element of the set. We hence have an orthogonality relation.

□

Theorem 6.0.16 (Wedderburn–Malcev)**.** *Let A be a finite dimentional $\mathbb{K}$-algebra such that $A/J(A)$ is a split algebra. Let $\pi : A \twoheadrightarrow A/J(A) : a \longmapsto \bar{a}$ be the quotient map.*

1. *There is a semsimple subalgebra $S \leq A$ such that $\pi|_S$ is an isomorphism. In other words there is a section $\sigma : A/J(A) \longrightarrow A$ of π.*
2. *If $T \leq A$ is another semisimple subalgebra such that $\pi|_T$ is an isomorphism, then T and S are conjugate. In other words, the section of π is unique up to conjugacy.*

Remark 6.0.17. Let A be a finite dimentional $\mathbb{K}$-algebra. Then A is *separable* if

1. A is semisimple,
2. consider the Wedderburn decomposition $A \cong \prod_i M_{n_i}(D_i)$, then $Z(D_i)/\mathbb{K}$ is a separable extension.

The Wedderburn–Malcev theorem also holds for algebra A such that $A/J(A)$ is separable.

Chapter 7

Symmetric algebras

7.1 Definition

Definition 7.1.1. Let A be a finite dimentional $\mathbb{K}$-algebra. Let M be a finitely generated left A-module. The *dual* of M is defined as $M^* = \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ endowed with the following structure of right A-module : if $f \in M^*$ and $a \in A$, then

$$\begin{aligned} fa : M &\longrightarrow \mathbb{K} \\ m &\longmapsto f(am). \end{aligned}$$

Proposition 7.1.2. *Let A be a finite dimentional $\mathbb{K}$-algebra. The following are equivalent :*

1. *There is an isomorphism of right A-modules $\phi : A_A \longrightarrow ({}_AA)^*$ which is symmetric : $\phi(a)(b) = \phi(b)(a)$, for every $a, b \in A$.*

2. *There exists a symmetric non degenerate bilinear form β on A that is symmetric and associative, i.e. such that $\beta(ab, c) = \beta(a, bc)$, for every $a, b, c \in A$.*

3. *There is a linear form $\lambda : A \longrightarrow \mathbb{K}$ which is symmetric, i.e. $\lambda(ab) = \lambda(ba)$, $\forall a, b \in A$, such that* $\ker\lambda$ *doesn't contain any right ideal of A.*

Proof.

1. $\Longleftrightarrow$ 2. From a symmetric $\phi : A_A \longrightarrow ({}_AA)^*$ we can construct a symmetric β defined as $\beta(a, b) = \phi(a)(b)$. Then, ϕ is an isomorphism if and only if β is non degenerate (as we are in finite dimentional vector spaces). The associativity is routine verifications.

2. $\Longleftrightarrow$ 3. If $\beta : A \times A \longrightarrow \mathbb{K}$ is symmetric associative, then define λ by $\lambda(a) = \beta(a, 1)$. Since β is associative, we have $\lambda(ab) = \beta(ab, 1) = \beta(a, b)$. Since β is symmetric, λ is too. Conversely, if λ is linear symmetric, then define $\beta(a, b) =$

$\lambda(ab)$. Since λ is symmetric, β is too. Moreover, β is associative because multiplication in A is too. Consider the following :

$$\begin{aligned} &\beta(a,x) = 0 && \forall x \in A \\ \iff &\lambda(ax) = 0 && \forall x \in A \\ \iff &aA \leq \ker\lambda \\ \iff &\ker\lambda \text{ cont. a right. ideal. cont. } a. \end{aligned}$$

Then it is clear that β is non degenerate if and only if $\ker\lambda$ doesn't contain any right ideal.

□

Definition 7.1.3 (Symmetric algebra)**.** Let A be a finite dimentional $\mathbb{K}$-algebra. Then A is called *symmetric* if one (and hence all) of the conditions of proposition 7.1.2 hold. If this case, the linear form λ is called a *symmetrising form*[1]. A symmetrizing form may not be unique.

Examples 7.1.4. 1. $A = M_n(\mathbb{K})$ is symmetric with symmetrizing form

$$\mathrm{tr} : M_n(\mathbb{K}) \longrightarrow \mathbb{K}.$$

Indeed, we know that tr is linear and that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$, $\forall X,Y \in M_n(\mathbb{K})$. Next, we have a canonical basis $\{E_{p,q}\}_{p,q}$ of $M_n(\mathbb{K})$. In order to prove that the associated bilinear form $\beta(X,Y) = \mathrm{tr}(X,Y)$ is non degenerate, it suffices to find a dual basis. Here it is $\{E_{q,p}\}_{p,q}$ because $E_{p,q}E_{r,s} = \delta_{q,r}E_{p,s}$, and therefore

$$\beta(E_{p,q}E_{r,s}) = \mathrm{tr}(E_{p,q}E_{r,s}) = \delta_{q,r}\delta_{p,s}.$$

Then clearly, β is non degenerate. Remark that a typical right ideal in $M_n(\mathbb{K})$ has form $\begin{pmatrix} 0 \\ \vdots \\ \mathbb{K}^n \\ \vdots \\ 0 \end{pmatrix} \not\leq \ker\mathrm{tr}.$

2. Let G be a finite group. The group algebra $\mathbb{K}G$ is symmetric with symmetrizing form

$$\begin{aligned} \lambda : \mathbb{K}G &\longrightarrow \mathbb{K} \\ g &\longmapsto \begin{cases} 1 & \text{if } g = 1 \\ 0 & \text{ow.} \end{cases} \end{aligned}$$

[1] "forme symmétrisante" in french

Then $\lambda(gh) = 1$ if and only if g and h are mutually inverse. Hence $\lambda(hg) = 1$ if and only if g and h are mutually inverse. By linearity, λ is symmetric. The corresponding bilinear form is given by

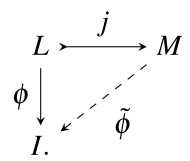
$$\begin{aligned} \beta : \mathbb{K}G \times \mathbb{K}G &\longrightarrow \mathbb{K} \\ (g,h) &\longmapsto \begin{cases} 1 & \text{if } g = h^{-1} \\ 0 & \text{ow.} \end{cases} \end{aligned}$$

Remark that $\{g^{-1}\}_{g\in G}$ is a dual basis of $\{g\}_{g\in G}$, and hence β is non degenerate.

7.2 Injective modules

Lemma 7.2.1. *Let R be a ring and let I be a finitely generated left R-module. The following are equivalent :*

1. *For any injective homomorphism $j : L \longrightarrow M$ between finitely generated R-modules, and for any homomorphism $\phi : L \longrightarrow I$, there exists lift $\tilde{\phi} : M \longrightarrow I$ such that the following diagram commutes :*

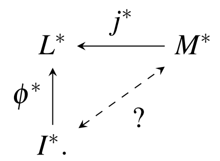


2. *Any injective homomorphism $I \longrightarrow M$ admit a retraction.*

Proof.

1. $\implies$ 2. Take $L = I$ and $\phi = \mathrm{id}_I$.

2. $\implies$ 1. With a huge loss of generality, we consider R to be a finite dimentional $\mathbb{K}$-algebra. Take L, M, j and ϕ as in point 1. and apply duality :



Then I^* is projective, j^* is surjective, and there exists $\widetilde{\phi^*}$ lifting ϕ^*. Since we consider finite dimentional modules over $\mathbb{K}$, dualization $(-)^*$ is involutive. Hence $\widetilde{\phi^*}^*$ lifts $(\phi^*)^* = \phi$.

□

Definition 7.2.2 (Injective module)**.** A R-module I satisfying one (and hence all) condition of lemma 7.2.1 is called an *injective module*.

Lemma 7.2.3. *Let M be a finitely generated left A-module. Then M is projective if and only if M^* is injective as a right A-module.*

Proof. Easy. □

Exercise 7.2.4. If A is a symmetric algebra, the isomorphism ${}_AA \cong (A_A)^*$ of left A-modules is also an isomorphism of right modules.

Proposition 7.2.5. *Let A be a finite dimentional symmetric $\mathbb{K}$-algebra. Then projective and injective modules coincide. We also say that A is* self-injective.

Proof. A_A is a free right A-module hence projective. Therefore $(A_A)^*$ is an injective left A-module. Since $(A_A)^* \cong {}_AA$ we have that ${}_AA$ is injective. Then $({}_AA)^{\oplus n}$ is injective (exercise), and any direct summand P of $({}_AA)^{\oplus n}$ is injective (exercise again). Hence all projective modules are injective. Dualize for the converse. □

Definition 7.2.6 (Socle of a module)**.** Let M be a finitely generated A-module. Then the *socle* of M, written $\operatorname{soc} M$ is the sum of all simple submodules of M. It is also the largest semisimple submodule of M.

Remark 7.2.7.

1. We know that indecomposable projective A-modules have a unique simple quotient. By duality, and using exercise 1 of sheet 12, we have that any indecomposable injective submodule have a unique simple submodule. Moreover, we have a bijection

$$\left\{ \begin{array}{c} \text{iso. classes of} \\ \text{indec. injective } A\text{-mods.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{iso. classes of} \\ \text{simple } A\text{-mods.} \end{array} \right\}$$

   which associates to an injective module I its socle $\operatorname{soc} I$.

2. If A is symmetric, then projective and injective modules coincide, and therefore any projective indecomposable module P have both a unique simple quotient $P/J(P)$ and a unique simple submodule $\operatorname{soc} P$.

Theorem 7.2.8. *Let A be a symmetric finite dimentional $\mathbb{K}$-algebra, and let P be an indecomposable projective A-module. Then $P/J(P) \cong \operatorname{soc} P$.*

Proof. We know that $P \cong Ae$, where e is a primitive idempotent of A, and that $P/J(P) = Ae/J(A)e$. The socle $\operatorname{soc} Ae$ is a left ideal of A, hence not contained in $\ker \lambda$, where $\lambda : A \longrightarrow \mathbb{K}$ is a symmetrizing form. Let $a \in \operatorname{soc} Ae$ be such that $\lambda(a) \neq 0$. We have $a = ae$, therefore $0 \neq \lambda(a) = \lambda(ae) = \lambda(ea)$, and $ea \neq 0$.

$$\begin{aligned} \phi : Ae &\longrightarrow \operatorname{soc} Ae \\ xe &\longmapsto xea. \end{aligned}$$

Clearly, ϕ is a homomorphism of left A-modules. It is non zero as $\phi(e) = ea \neq 0$. As $\mathrm{soc} Ae$ is simple, we have that ϕ is surjective. Moreover, ϕ induces an isomorphism $Ae/\ker\phi \cong \mathrm{soc} Ae$. As Ae have a unique simple quotient, we have that $Ae/J(A)e \cong Ae/\ker\phi$, and so $P/J(P) \cong \mathrm{soc} P$. □

Example 7.2.9. Let Q be a quiver without oriented cycles, so $\mathbb{K}Q$ is finite dimentional.

- Trivial case : no arrow. Then $\mathbb{K}Q \cong \mathbb{K}^n$, where n is the number of vertex of Q. Hence $\mathbb{K}Q$ is split semisimple, hence symmetric (exercise 5.a, sheet 12).

- Non trivial cases : there is at least one arrow. We want to prove that $\mathbb{K}Q$ isn't symmetric. Let v be the target of an arrow, and l_v be the empty path at v, which is a primitive idempotent of $\mathbb{K}Q$. Then $P_v = \mathbb{K}Ql_v$ is projective indecomposable (in fact, it is spanned by all paths ending at v). Also, $P_v/J(P_v) = \mathbb{K}Ql_v/J(\mathbb{K}Q)l_v$ is a one dimentional simple module generated by the class of l_v.

  Let u be the origin of a maximal path π ending at v. Then there is no arrow with target u be maximality. Therefore $\mathbb{K}Ql_u = \mathbb{K}l_u$ is simple. There is a homomorphism of left $\mathbb{K}Q$-modules

$$\begin{aligned} m_\pi : \mathbb{K}Ql_u = \mathbb{K}l_u &\longrightarrow \mathbb{K}Ql_v \\ l_u &\longmapsto l_u\pi. \end{aligned}$$

  Since $\mathbb{K}l_u$ is one dimentional and $l_u\pi \neq 0$, we have that m_π is injective. Therefore the simple module $\mathbb{K}Ql_u = \mathbb{K}l_u$ corresponding to l_u is isomorphic to a submodule of the projective module $\mathbb{K}Ql_v$ corresponding to l_v. Hence $\mathbb{K}Ql_v$ has a simple submodule $\mathbb{K}l_u$ in its socle, and this simple submodule is not isomorphic to $\mathbb{K}Ql_v/J(\mathbb{K}Ql_v)$. By the previous theorem, $\mathbb{K}Q$ is not symmetric.

Chapter 8

Finite representation type

8.1 Definition

Definition 8.1.1. Let A be a finite dimentional $\mathbb{K}$-algebra. We say that A has *finite representation type* if there are finitely many isomorphism classes of finitely generated indecomposable left A-modules.

Lemma 8.1.2. *$A = \mathbb{K}[X]/(X^n)$ is symmetric, where $n \geq 1$. In particular, ${}_AA$ is an injective module.*

Proof. Let x be the class of X in A, so $x^n = 0$. Obviously, $1, x, \ldots, x^{n-1}$ is a $\mathbb{K}$-basis of A. We claim that the only ideals of A are Ax^i, for $0 \leq i \leq n$. Indeed, an ideal I of A has the form $I = J/(X^n)$, where J is an ideal of $\mathbb{K}[X]$ containing X^n. But $\mathbb{K}[X]$ is a PID, and so $J = (f)$, for $f \in \mathbb{K}[X]$ monic, and $f | X^n$. Hence, $f = X^i$ for $0 \leq i \leq n$, which proves the claim. Define

$$\begin{aligned} \lambda : A &\longrightarrow \mathbb{K} \\ x^i &\longmapsto 1. \end{aligned}$$

Clearly, λ is symmetric, and $\ker \lambda$ doesn't not contain any nonzero ideal of A. □

Remark 8.1.3. The algebra $A = \mathbb{K}[X]/(X^n)$ is called the *algebra of truncated polynomials*.

Theorem 8.1.4. *$A = \mathbb{K}[X]/(X^n)$ has finite representation type. More precisely :*

1. *the only indecomposable A-modules (up to isomorphism) are A/Ax^i, for $1 \leq i \leq n$;*

2. *A is* uniserial*, i.e. any module has a unique composition series.*

Proof. By induction on n. If $n = 1$, then $A \cong \mathbb{K}$, and the result is obvious. Suppose now $n \geq 2$, and let M be a finitely generated A-module.

Suppose that there exists $m_1 \in M$ such that $x^{n-1} m_1 \neq 0$. Then m_1 generates a submodule Am_1 with basis $m_1, xm_1, \ldots, x^{n-1} m_1$. Indeed, if $\sum_{i=0}^{n-1} \lambda_i x^i m_1 = 0$, then

$x^{n-1}\sum_{i=0}^{n-1}\lambda_i x^i m_1 = \lambda_0 x^{n-1} m_1 = 0$, and so $\lambda_0 = 0$, and repeat with x^{n-2}, etc. Therefore Am_1 is a free module isomorphic to ${}_AA$, which is itself injective (by the previous lemma). Hence Am_1 is injective as well, and the injection $Am_1 \hookrightarrow M$ has a retraction, hence $M = Am_1 \oplus M_2$. Suppose that there exists $m_2 \in M_2$ such that $x^{n-1}m_2 \neq 0$. Then by the same argument, we have $M = \underbrace{Am_1}_{\cong A} \oplus \underbrace{Am_2}_{\cong A} \oplus M_3$. Continuing in this way until the initial assumption is false leads to a decomposition

$$M \cong Am_1 \oplus \cdots \oplus Am_k \oplus N,$$

where N is a submodule such that $x^{n-1}N = 0$.

Hence N can be viewed as a B-module, where $B = A/Ax^{n-1} \cong K[X]/(X^{n-1})$. By induction, the only indecomposable B-modules are B/Bx^i, for $1 \leq i \leq n-1$. Therefore $N \cong \bigoplus_i (B/Bx^i)^{\oplus k_i}$. Clearly, $B/Bx^i \cong (A/A^{n-1})/(Ax_i/Ax^{n-1}) \cong A/Ax^i$. We view B/x^iB as a left A-module (with x_{n-1}) acting by 0.

This proves that M decomposes as a direct sum of modules of the form A/Ax^i. We need to show that each module A/Ax^i is indeed indecomposable. We know that $\mathrm{End}_R R \cong R^{\mathsf{op}}$, for any ring R. In particular, we have an isomorphism

$$\begin{aligned} \mathrm{End}_A(A/Ax_i) &\longleftrightarrow (A/Ax_i)^{\mathsf{op}} = A/Ax_i \\ f &\longmapsto f(\bar{1}) \\ m_{\bar{a}} &\longleftarrow\!\shortmid \bar{a}. \end{aligned}$$

Now $A/Ax_i \cong \mathbb{K}[X]/(X^i)$ is a local ring with maximal ideal Ax/Ax^i because the only maximal ideal of $K[X]$ containing X^i is (X). Hence, since $\mathrm{End}_A(A/Ax_i)$ is local, we have that A/Ax_i is indecomposable. This proves that A has finite representation type, with n indecomposable modules (up to isomorphism).

For uniseriality[1], remark that the only ideals of $\mathbb{K}[X]$ containing X^n are $(X^n) > \cdot > (X)$. Therefore, the only ideals of $A = \mathbb{K}[X]/(X^n)$ are $A > Ax > \cdots > Ax^{n-1} > 0$. This proves that ${}_AA$ is uniserial, and so are any of its quotients A/Ax_i.

□

Remark 8.1.5. Ax^i is an indecomposable submodule of ${}_AA$. It is in fact it is isomorphic to A/Ax^{n-i} with

$$\begin{aligned} A/Ax^{n-i} &\longrightarrow Ax^i \\ \bar{1} &\longmapsto x^i \\ \bar{x} &\longmapsto x^{i+1} \\ &\vdots \\ \bar{x}^{n-i} &\longmapsto x^n = 0. \end{aligned}$$

[1] is this even a real word ?

8.2 Group algebras of finite representation type

Let G be a finite group. Then $A = \mathbb{K}G$ is finite dimentional.

1. If $\operatorname{char}\mathbb{K} = 0$ or $\operatorname{char}\mathbb{K} = p$ and $p \not| |G|$, then by Maschke theroem, $\mathbb{K}G$ is semisimple. Then every indecomposable module is simple and in particular, there are finitely many of them, up to isomorphism. Thus $\mathbb{K}G$ is of finite representation type.

2. If $\operatorname{char}\mathbb{K} = p$, and if G is a p-group, i.e. $|G| = p^n$ for some $n \in \mathbb{N}^*$, then we obtain theorem 8.2.2.

3. If $\operatorname{char}\mathbb{K} = p$ which divides $|G|$, then some standard method called induction and restriction allow to pass from a p-Sylow subgroup of G to the whole of G, and we obtain theorem 8.2.3.

Lemma 8.2.1. *Let G be a finite p-group, with p prime.*

1. *Any maximal subgroup of G is normal and has index p.*

2. *If there is a unique maximal subgroup, then G is cyclic.*

3. *If there is at least two distinct maximal subgoups, then G has a quotien isomorphic to C_p^2.*

Proof. 1. By induction on n, where $|G| = p^n$. If $n = 1$, then G is cyclic and 1 is the only maximal subgroup, which of course has index p. Suppose $n \geq 2$, and let M be a maximal subgroup. We use the fact that $Z = Z(G)$ is non trivial (consequence of the class equation). We have two cases :

   (a) $Z \leq M$, then M/Z is a maximal subgroup of G/Z. Since $|G/Z| < |G|$, induction tells us that M/Z is normal, and has index p. It follows that M is normal in G with index p.

   (b) $Z \not\leq M$, then $M < ZM$, and the later is a subgroup since Z is normal. So $ZM = G$ by maximality of M. Let $g \in G$, then $g = zm$ with $z \in Z$, $m \in M$. Then $gMg^{-1} = zmMm^{-1}z^{-1} = zMz^{-1} = M$. Hence M is normal. Moreover $GM = H$ is a group without any subgroup apart from 1 and H. Hence $H \cong C_p$ and M has index p

2. Suppose M is the unique maximal subgroup of G. Let $g \in G \setminus M$. Then (g) is not contained in M, hence not contained in any maximal subgoup, hence $(g) = G$, and G is cyclic.

3. Suppose that M_1 and M_2 are two distince maximal subgroups of G. Then $M_1 < M_1M_2 \leq G$, and by maximality of M_1 we have $M_1M_2 = G$. Let $N = M_1 \cap M_2$. By the second isomorphism theorem, we have $G/M_1 \cong M_2/N$ and similarily, $G/M_2 \cong M_1/N$. The obvious group homomorphism $G \longrightarrow$

$G/M_1 \times G/M_2 \cong C_p^2$ has kernel N. Therefore it induces an injective map $G/N \longrightarrow C_p^2$. However $|G/N| = p^2$ and so $G/N \cong C_p^2$.

□

Theorem 8.2.2. *With the assumptions of point 2., i.e.* $\operatorname{char}\mathbb{K} = p$ *and G is a p-group, $\mathbb{K}G$ has finite representation type if and only if G is cyclic. More precisely,*

1. *If G is cyclic, then $\mathbb{K}G$ is uniserial and there are $|G|$ indecomposable modules up to isomorphism.*

2. *If G is not cyclic, then $\mathbb{K}G$ has a quotient isomorphic to $\mathbb{K}[X,Y]/(X^2,Y^2,XY)$, which has infinite representation type.*

Proof. 1. If G is cyclic, let $x = g - 1 \in \mathbb{K}G$, where g is a generator of G, and define

$$\begin{aligned} \phi : \mathbb{K}[X] &\longrightarrow \mathbb{K}G \\ X &\longmapsto x = g - 1. \end{aligned}$$

It is a surjective algebra homomorphism because $\phi(X+1) = g$, $\phi((X+1)^k) = g^k$, and $\operatorname{im}\phi$ contains a basis of $\mathbb{K}G$. Now $x^{p^n} = (g-1)^{p^n} = g^{p^n} - 1 = 0$, so $\ker\phi \geq (X^{p^n})$, and ϕ induces a surjective algebra homomorphism $A = \mathbb{K}[X]/(X^{p^n}) \longrightarrow \mathbb{K}G$. We have $\dim_{\mathbb{K}} A = \dim_{\mathbb{K}} \mathbb{K}G = p^n$, so $A \cong \mathbb{K}G$. We know by theorem 8.1.4 that A is uniserial and of finite representation type.

2. Suppose G not cyclic. By lemma 8.2.1, there is a normal subgroup $N < G$ such that $G/N \cong C_p^2$. Therefore there is a surjective algebra homomorphism $\phi : \mathbb{K}G \longrightarrow \mathbb{K}(C_p^2)$, and we have an isomorphism (same argument as before)

$$\begin{aligned} \mathbb{K}[X,Y]/(X^p,Y^p) &\longrightarrow \mathbb{K}(C_p^2) \\ X &\longmapsto g - 1 \\ Y &\longmapsto h - 1, \end{aligned}$$

where g is a generator of $C_p \times 1$ and h is a generator of $1 \times C_p$. Now has a quotient $B = \mathbb{K}[X,Y]/(X^2,Y^2,XY)$. So $\mathbb{K}G$ has a quotient isomorphic to B. It is shown in example 4.2.6 that B has infinite representation type, provided that $\mathbb{K}$ is infinite. If $\mathbb{K}$ is finite, then the same result hold (see exercise 6 of sheet 13). It follows that $\mathbb{K}G$ also have infinite representation type (because any indecomposable module of a quotient $\mathbb{K}G/I$ remains indecomposable seen as a module over the base ring, on which I acts as 0).

□

Theorem 8.2.3. *With the assumptions of point 3., i.e.* $\operatorname{char}\mathbb{K} = p \mid |G|$, *$\mathbb{K}G$ has finite representation type if and only if the p-Sylow subgroup of G are cyclic.*

Proof. Not treated in this course. □

8.3 Quivers of finite representation type

Let Q be a finite quiver. Suppose that Q has no oriented cycles, so that $\mathbb{K}Q$ is finite dimentional. Associated with Q there is an unoriented graph $\bar{Q}$ which is Q with arrows replaced by unoriented edges.

Theorem 8.3.1 (Gabriel, 1972)**.** *$\mathbb{K}Q$ has finite representation type if and only if the undirected graph $\bar{Q}$ is a disjoint union of Dynkin graph.*

Proof. Not treated in this course. □

This concludes this course about finite dimentional algebra, and is also the last course of prof. J. Thévenaz given to math students !

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