WTF ARE MONADS WITH ARITIES AND PARAMETRIC RIGHT ADJOINTS?

CHT

> A monad with arity allows a powerful nerve theorem. A parametric right adjoint can be constructed from its local left adjoint with very minimal data.
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1. Monads with arities

Let $\mathcal{C}$ be a category with a dense generator $i : \Theta_0 \hookrightarrow \mathcal{C}$. Note that we have a nerve $N_0 : \mathcal{C} \longrightarrow \widehat{\Theta_0}$ mapping $c \in \mathcal{C}$ to the presheaf $\mathcal{C}(i(-), c)$. Since i is dense, N_0 is fully faithful (in fact, the converse also hold).

A monad T on $\mathcal{C}$ *has arities in* Θ_0 if $T : \mathcal{C} \longrightarrow \widehat{\Theta_0}$ takes the canonical Θ_0-cocones to colimit cocones in $\widehat{\Theta_0}$. Explicitely, if $c \in \mathcal{C}$, then the canonical Θ_0-cocone of c is the natural transformation on the left, and the arity condition states that the cocone on the right is colimiting:

$$\Theta_0/c \;\underset{c}{\overset{}{\Downarrow}}\!\alpha\; \mathcal{C}, \qquad \Theta_0/c \;\underset{c}{\Downarrow}\!\alpha\; \mathcal{C} \xrightarrow{T} \mathcal{C} \xrightarrow{N_0} \widehat{\Theta_0}.$$

Equivalently, the following triangle exhibits $N_0 T$ as the left Kan extension of $N_0 T i$ along i:

$$\begin{array}{ccccccc} \Theta_0 & \xrightarrow{i} & \mathcal{C} & \xrightarrow{T} & \mathcal{C} & \xrightarrow{N_0} & \widehat{\Theta_0} \\ {\scriptstyle i}\downarrow & & & & & \nearrow_{T} & \\ \mathcal{C} & & & & & & \end{array}$$

i.e. $N_0 T c = \operatorname{colim}_{\theta \to c} N_0 T \theta$.

Let $\mathcal{C}^T$ be the Eilenberg–Moore category of T. We abuse notations by letting $T : \mathcal{C} \longrightarrow \mathcal{C}^T$ be the free algebra functor. Let $i_T : \Theta_T \hookrightarrow \mathcal{C}^T$ the full subcategory spanned by $T\Theta_0$. As for i, we obtain a nerve $N_T : \mathcal{C}^T \longrightarrow \widehat{\Theta_T}$. The situation is summarized in the following squares

$$\begin{array}{ccc} \Theta_T & \xrightarrow{i_T} & \mathcal{C}^T \\ {\scriptstyle T}\uparrow & & \uparrow{\scriptstyle T} \\ \Theta_0 & \xrightarrow{i} & \mathcal{C}, \end{array} \qquad \begin{array}{ccc} \mathcal{C}^T & \xrightarrow{N_T} & \widehat{\Theta_T} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle -|_{\Theta_0}} \\ \mathcal{C} & \xrightarrow{N_0} & \widehat{\Theta_0}. \end{array}$$

The left one commutes, but in general, the right one does not.

Theorem 1.1 (Nerve theorem)**.**

1. *The subcategory Θ_T is a dense generator of $\mathcal{C}^T$. Equivalently, N_T is fully faithful.*
2. *A presheaf $X \in \widehat{\Theta_T}$ is in the essential image of N_T if and only if it satisfies the* Segal condition*, namely, if $X|_{\Theta_0} \in \widehat{\Theta_0}$ (that is, the restriction of X to free morphisms) is in the essential image of N_0*

Corollary 1.2 (Leinster)**.** *A presheaf $X \in \widehat{\Theta_T}$ is Segal if and only if for all $\theta \in \Theta_0$, X maps the canonical Θ_0-cocone of θ to a limit cone in* $\mathcal{S}\mathrm{et}$*, i.e. the cone on the right is limiting:*

$$\Theta_0/\theta \;\underset{\theta}{\Downarrow}\!\alpha\; \Theta_0, \qquad \Theta_0^{\mathrm{op}}/\theta \;\underset{\theta}{\Uparrow}\!\alpha\; \Theta_0^{\mathrm{op}} \xrightarrow{X} \mathcal{S}\mathrm{et}.$$

Example 1.3. Let $\Theta_0 = \mathbb{\Delta}$, $\mathcal{C} = \widehat{\mathbb{\Delta}}$, and T free category monad, i.e. that which arises from the adjunction $h : \widehat{\mathbb{\Delta}} \overset{\dashv}{\longleftrightarrow} \mathcal{C}\mathrm{at} : N$. For $[n] \in \mathbb{\Delta}$, $T\Delta[n] = Nh\Delta[n] = N[n] = \Delta[n]$, and in particular, $\Theta_T = \mathbb{\Delta}$. The colimit condition is trivially satisfied for T to have arities in $\mathbb{\Delta}$. The nerve theorem then states that

1. $\mathbb{\Delta}$ is a dense generator of $\widehat{\mathbb{\Delta}}^T = \mathcal{C}\mathrm{at}$, and that $N : \mathcal{C}\mathrm{at} \longrightarrow \widehat{\mathbb{\Delta}}$ is fully faithful;
2. since $\Theta_0 = \Theta_T$, the Segal condition doesn't say anything in this case.

Leinster's criterion states that $X \in \widehat{\mathbb{\Delta}}$ is a nerve if and only if it maps canonical cocones of the form $\mathbb{\Delta}/[n]$ to limit cones in $\mathcal{S}\mathrm{et}$, which is the usual Segal condition for simplicial sets.

2. Parametric right adjoints

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor, where $\mathcal{C}$ has a terminal object 1. Then F factors as

$$\mathcal{C} = \mathcal{C}/1 \xrightarrow{F_1} \mathcal{D}/F1 \longrightarrow \mathcal{D}.$$

We say that F is a *parametric right adjoint* (p.r.a. for short) if F_1 has a left adjoint L_F.

Example 2.1. A polynomial functor $P = t_! p_* s^*$

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

is a p.r.a., where $P_1 = p_* s^*$ and $L_P = s_! p^*$.

A morphism $f : D \longrightarrow FC$ in $\mathcal{D}$ is said F*-generic* if for every solid square

$$\begin{array}{ccc} D & \xrightarrow{x} & FA \\ {\scriptstyle f}\downarrow & \overset{Ft}{\nearrow} & \downarrow{\scriptstyle Fy} \\ FC & \xrightarrow{Fz} & FB \end{array}$$

there exist a unique morphism $t : C \longrightarrow A$ filling the diagram.

Theorem 2.2 ([Web07])**.** *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor, where $\mathcal{C}$ has a terminal object 1. The following are equivalent:*

1. *F is a p.r.a., meaning the sliced functor $F_1 : \mathcal{C}/1 \longrightarrow \mathcal{D}/F1$ has a left adjoint;*
2. *for all $C \in \mathcal{C}$, the sliced functor $F : \mathcal{C}/C \longrightarrow \mathcal{D}/FC$ has a left adjoint;*
3. *any morphism $f : D \longrightarrow FC$ has a generic-free factorization, i.e. f factors as*
$$D \xrightarrow{g} FB \xrightarrow{Fh} FC$$
where g is F-generic.

Theorem 2.3 ([Web07])**.** *For $F : \widehat{\mathcal{A}} \longrightarrow \widehat{\mathcal{B}}$, the following are equivalent:*

1. *F is a p.r.a.;*
2. *any morphism $f : b \longrightarrow F1$ in $\widehat{\mathcal{B}}$ has a generic-free factorization.*

If $F : \widehat{\mathcal{A}} \longrightarrow \widehat{\mathcal{B}}$ is a p.r.a. between presheaf categories, then the left adjoint $L_F : \widehat{\mathcal{B}}/F1 \cong \widehat{\mathcal{B}/F1} \longrightarrow \widehat{\mathcal{A}}$ can be defined as follows. For an element $f : b \longrightarrow F1$ of $F1$, let $L_F f$ be the object sitting in the middle of the generic-free factorization of f:
$$b \xrightarrow{\text{generic}} FL_F f \xrightarrow{\text{free}} F1.$$
Then, extend L_F by left Kan extension along the Yoneda embedding to get $L_F : \widehat{\mathcal{B}/F1} \longrightarrow \widehat{\mathcal{A}}$. Since F_1 is right adjoint of L_F, the classical nerve formula gives
$$FX_b = \sum_{x \in F1_b} \widehat{\mathcal{A}}(L_F x, X)$$
for $X \in \widehat{\mathcal{A}}$. Note that conversely, F is completely determined by a choice of presheaf $F1 \in \widehat{\mathcal{B}}$ and a functor $\mathcal{B}/F1 \longrightarrow \widehat{\mathcal{A}}$.

Let now $T : \widehat{\mathcal{A}} \longrightarrow \widehat{\mathcal{A}}$ be a p.r.a. monad, and Θ_0 be the full subcategory of $\widehat{\mathcal{A}}$ spanned by the image of $L_T : \mathcal{A}/T1 \longrightarrow \widehat{\mathcal{A}}$. Elements isomorphic to some element of Θ_0 are called *T-cardinals*.

Proposition 2.4 ([Web07], propositions 4.20 and 4.22)**.**

1. *Representables are T-cardinals.*
2. *If p is a T-cardinal and $f : p \longrightarrow Tq$ is T-generic, then q is a T-cardinal.*
3. *The monad T has arities in Θ_0.*

Example 2.5. Consider $\mathcal{G}\mathrm{raph} = \widehat{[0] \rightrightarrows [1]}$, the category of graphs, and T the free category monad. Clearly, a moorphism $f : [1] \longrightarrow TG$ is the data of a path in G, and if it has length n, one can check that the following is a generic-free factorization of f:
$$1 \xrightarrow{\text{endpoints}} T[n] \xrightarrow{\text{the path}} TG.$$
Consequently, all the $[n]$ are T-cardinals. On the other hand, a morphism $[0] \longrightarrow TH$ is T-generic if and only if $H \cong [0]$, whence the T-cardinals are exactly the linear graphs $[n]$. Therefore, T has arites in Θ_0, the full subcategory of $\mathcal{G}\mathrm{raph}$ spanned by the $[n]$'s, and $\Theta_T \simeq \mathbb{\Delta}$.

Example 2.6. For $\widehat{\mathcal{M}}$ the category of multigraphs, where $\mathcal{M}$ is the adequate shape theory, and T the free symmetric multicategory monad, we have $\Theta_T \simeq \Omega$, the category of dendrices of [MW07]. 早っ!

References

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