WTF IS KOSZUL DUALITY?

CHT

Did you know you could kill someone by dropping [LV12] *on its head?*

1. Fiber bundles

A *fiber bundle* [Hat02] (also called *locally trivial fiber bundle*) $F \longrightarrow E \xrightarrow{p} B$ is a "short exact sequence of spaces", more precisely, it is a continuous map p such that every point $b \in B$ has an open neighborhood $b \in U$ for which there is a homeomorphism $h : p^{-1}(U) \longrightarrow U \times F$, called *local trivialization*, making the following triangle commute

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{h} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{proj}} & \\ & U. & \end{array}$$

The space F is called the *fiber*, E is the *total space*, and B is the *base space*. Examples include *trivial bundles* $F \longrightarrow F \times B \longrightarrow B$, but also the Möbius strip $D^1 \longrightarrow M \longrightarrow S^1$, the Klein bottle $S^1 \longrightarrow K^2 \longrightarrow S^1$, and the Hopf fibrations. Also, a fiber bundle with discrete fiber is a covering space.

Theorem 1.1. *Let $F \longrightarrow E \xrightarrow{p} B$ be a fiber bundle. Then $p_* : \pi_n(E, F) \longrightarrow \pi_n B$ is an isomorphism (basepoints are left implicit). Thus, if B is path-connected, the long relative exact sequence becomes*

$$\cdots \to \pi_{n+1} B \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \cdots .$$

2. Simplicial twisting maps

This section quickly surveys [May92], and we only deal with simplicial sets.

A map $p : E \longrightarrow B$ of simplicial sets is called a *fiber bundle of fiber F* if for every cell $b : \Delta[n] \longrightarrow B$, the projection $\Delta[n] \times_B E \longrightarrow \Delta[n]$ of the following pullback

$$\begin{array}{ccc} \Delta[n] \times_B E & \longrightarrow & E \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ \Delta[n] & \xrightarrow{b} & B \end{array}$$

is isomorphic to the projection $\Delta[n] \times F$. If F is a Kan complex, then p is a Kan fibration. The bundle is said trivial if $E = F \times B$ and p is just the projection.

Let G be a simplicial group (thus a Kan complex) acting on the left on E, which we write $G \circlearrowleft E$. The action is said *effective* if $\forall g \in G$, if g acts like the unit 1, then $g = 1$; it is said *principal* if $\forall g \in G$, $\forall x \in E$, if $gx = x$, then $g = 1$.

If $G \circlearrowleft E$ principally, then we obtain what we call a *principal fiber bundle* $F \longrightarrow E \xrightarrow{p} B = G \backslash E$. It seems clear that it is a fiber bundle, but I could not find that statement anywhere... A *cross section* (or just *section*) of p is a map $\sigma : B \longrightarrow E$ such that $p\sigma = \mathrm{id}_B$. It exists only if the bundle is trivial. Thus we relax the constraints on σ and allow it to not preserve the 0-th face operator d_0, and call such a map a *pseudo cross section* (or just *pseudo section*).

The following way of relating twisted cartesian products and twisting map is taken from [Pro11], which is itself based on [May92]. Note that for such a pseudo cross section σ, and $b \in B$,

$$p\sigma d_0(b) = d_0(b) = d_0 p\sigma(b) = p d_0 \sigma(b)$$

and thus $\sigma d_0(b)$ and $d_0\sigma(b)$ lie in the same fiber of p. Since the action of G is principal, there exist a unique $\tau(b) \in G$ such that $d_0\sigma(b) = \sigma d_0(b) \cdot \tau(b)$. The map $\tau : B \longrightarrow G$ of degree -1 (meaning $\tau B_n \subseteq G_{n-1}$) satisfies the following properties

$$\begin{aligned} d_0\tau(b) &= (\tau d_0(b))^{-1} \cdot \tau d_1(b), & \\ d_i\tau(b) &= \tau d_{i+1}(b) & i \geq 1, \\ s_i\tau(b) &= \tau s_{i+1}(b) & i \geq 0, \\ \tau s_0(b) &= 1. & \end{aligned}$$

A map of degree -1 satisfying those properties is called a *twisting map*. Interestingly, τ is enough to recover the pseudo section σ. The *twisted cartesial product* $E = B \times_\tau G$ is the usual cartesian product $B \times G$, but where the 0-th face operator has been replaced by

$$d_0(b, g) = (d_0(b), \tau(b) \cdot d_0(g)),$$

the fibration $p : E \longrightarrow B$ maps (b, g) to b, and the pseudo section σ maps b to $(b, 1)$. This construction exhibits a bijection between cross section of principal G-bundles over B and twisting map $B \longrightarrow G$.

2.1. Simplicial bar and cobar.

The $\mathfrak{W}$ and $\mathfrak{G}$ constructions presented here are covered in [May92] and briefly discussed in [Pro11].

Let G be a simplicial group, and consider the category $\mathcal{B}_G$ of tuples (X, τ), where X is a simplicial set and $\tau : X \longrightarrow G$ is a twisting map. Then $\mathcal{B}_G$ has a terminal object $\mathfrak{W}(G) \longrightarrow G$ which may be thought as a "bar construction on G".

Dually, if X is a reduced ($X_0 = \{*\}$) simplicial set, let $\mathcal{C}_X$ be the category of tuples (G, τ), where $\tau : X \longrightarrow G$ is a twisting map. Then $\mathcal{C}_X$ has an initial object $X \longrightarrow \mathfrak{G}(X)$, which may be thought as a "cobar construction on X".

Theorem 2.1. *The canonical morphism $GW(G) \longrightarrow G$ and $X \longrightarrow WG(X)$ are homotopy equivalences.*

3. Homological twisting maps

3.1. Intermezzo in derived algebra.

See [Yek20]. A dg-ring $\Bbbk$ is a graded algebra ($\Bbbk_i \Bbbk_j \subseteq \Bbbk_{i+j}$) over a ground ring, with a differential ∂ satisfying the graded Leibniz rule

$$\partial(ab) = \partial(a)b + (-1)^{|a||b|} a\partial(b).$$

It is said weakly commutative if $ab = (-1)^{|a||b|}ba$, and strongly commutative if it is weakly commutative, and if $aa = 0$ for all odd a (i.e. $|a|$ odd). If the ground ring is not of caracteristic 2, the two notions are equivalent. Any ring can be seen as a dg-ring concentrated in degree 0.

A $\Bbbk$-dg-module is a *dg-module* $M = \bigoplus_i M_i$ with a graded action $\Bbbk_i M_j \subseteq M_{i+j}$ satisfying

$$\partial(am) = \partial(a)m + (-1)^{|a|}a\partial(m).$$

Let $dg\mathcal{A}\mathrm{lg}_\Bbbk$ be the category of differential graded $\Bbbk$-algebras. The tensor product of $M, N \in dg\mathcal{A}\mathrm{lg}_\Bbbk$ is given by

$$(M \otimes N)_n = \bigoplus_{i+j=n} M_i \otimes N_j$$

with differential

$$\partial(m \otimes n) = \partial(m) \otimes n + (-1)^{|m|} m \otimes \partial(n).$$

The inner hom is given by

$$\hom(M, N)_n = \bigoplus_i \mathcal{A}\mathrm{lg}_\Bbbk(M_i, N_{i+n})$$

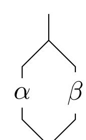
with differential

$$\partial(\phi) = \partial \circ \phi - (-1)^{|\phi|}\phi \circ \partial.$$

Note the minus in the middle, and that ϕ is an even cocycle if and only if $\partial \circ \phi = \phi \circ \partial$, that is, if it is a morphism of dg-algebras.

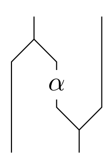
The following material can be found in [May92, Pro11], and in [LV12], although expressed differently. Let C be a dg-coalgebra with comultiplication D, and A be a dg-algebra with multiplication μ, both over some ground commutative ring $\Bbbk$. In particular, C and A are dg-modules over $\Bbbk$ and so we have a dg-module $\hom(C, A)$ defined previously. We endow it with a structure of dg-algebra with the following *cup product* (called *convolution product* and denoted by $\star$ in [LV12]):

$$\alpha \smile \beta = \mu(\alpha \otimes \beta)D$$



If M is a right C-dg-comodule, and N a left A-dg-module, then the $\Bbbk$-tensor $M \otimes N$ has a structure of left $\hom(C, A)$-dg-module with the following *cap product*

$$\alpha \frown (m \otimes n) = (1 \otimes \mu)(1 \otimes \alpha \otimes 1)(D \otimes 1)(m \otimes n)$$



Let $t : C \longrightarrow A$ be a morphism of degree -1. The *twisted tensor* $M \otimes_t N$ is defined as $M \otimes N$ with endowed with the following map of degree -1:

$$\partial_t(m \otimes n) = \partial(m \otimes n) + t \frown (m \otimes n).$$

Of course ∂_t has no reason to be a differential in general, but it is if and only if t satisfies the Cartan–Maurer relation [LV12]

$$\partial(t) + t \smile t = 0,$$

and in this case, we call t a *twisting morphism* (or *twisting cochain*, *Brown cochain* [Pro11]). Recall that the differential $\partial(t)$ of t in $\hom(C, A)$ is given by $\partial \circ t + t \circ \partial$ since again, $|t| = -1$.

4. Bar and cobar

Let R be a ring and C a chain complex in ${}_R\mathcal{M}\mathrm{od}$. Write $\blacktriangle^n C$ for the shifted complex where $\blacktriangle^n C_i = C_{i-n}$. The litterature writes $C(-n)$ [Wu16], $s^n C$ [LV12], $\uparrow^n C$ [Pro11], $C[n]$. The $\blacktriangle^n$ notation is prefered with n is positive; if it is negative, we let $\blacktriangledown^n = \blacktriangle^{-n}$.

Let A be a dg-algebra over $\Bbbk$, and $\mathcal{B}_A$ be the category of twisting morphisms over A, i.e. the category of tuples (C, τ), where $\tau : C \longrightarrow A$ is a twisting morphisms, and where morphisms are adequatly commuting dg-coalgebra morphisms. Then $\mathcal{B}_A$ has a terminal object $\beta : \mathrm{B}(A) \longrightarrow A$, the *bar construction* of A which we now define. As a graded algebra, we let

$$\mathrm{B}(A) = T(\blacktriangle\bar{A})$$

with the deconcatenation product, where $\bar{A} = A/\Bbbk$. It is customary to denote a homogeneous element $\blacktriangle a_1 \otimes \cdots \otimes \blacktriangle a_k$ of $\mathrm{B}(A)$ as $[a_1|\cdots|a_k]$. The twisting morphism β is the composite

$$\mathrm{B}(A) = T(\blacktriangle\bar{A}) \xrightarrow{\text{proj}} \blacktriangle\bar{A} \xrightarrow{\blacktriangledown} \bar{A} \longrightarrow A,$$

explicitly

$$\beta(1) = 0, \quad \beta[a] = a, \quad \beta[a_1|\cdots|a_k] = 0 \text{ for } k > 1.$$

The differential on $\mathrm{B}(A)$ is then specifically crafted for β to indeed be a twisting morphism:

$$\partial_k = \sum_{i=1}^{k-1} \mathrm{id}^{\otimes i-1} \otimes \mu \otimes \mathrm{id}^{\otimes k-i-1} + \sum_{i=1}^{k} \mathrm{id}^{\otimes i-1} \otimes \partial \otimes \mathrm{id}^{\otimes k-i}$$

where μ is the multiplication of A. Explicitly:

$$\partial[a_1|\cdots|a_k] = \sum_{i=1}^{k-1} (-1)^{i+|a_1|+\cdots+|a_i|}[a_1|\cdots|\mu(a_i, a_{i+1})|\cdots|a_k] + \sum_{i=1}^{k} (-1)^{i-1+|a_1|+\cdots+|a_{i-1}|}[a_1|\cdots|\partial(a_i)|\cdots|a_k].$$

Recall that elements in brackets are implicitely shifted.

Dually, for C a dg-coalgebra, let $\mathcal{C}_C$ be the category of twisting morphisms under C, i.e. tuples (A, τ) where $\tau : C \longrightarrow A$ is a twisting morphisms, and where morphisms are adequatly commuting dg-algebra morphisms. Then $\mathcal{C}_C$ has an initial object $\omega : C \longrightarrow \Omega(C)$, the *cobar construction* of C, wich we now define. As a graded algebra, we let

$$\Omega(C) = T(\blacktriangledown\bar{C})$$

with the concatenation product, where $\bar{C} = C/\Bbbk$. It is customary to denote a homogeneous element $\blacktriangledown c_1 \otimes \cdots \otimes \blacktriangledown c_k$ of $\Omega(C)$ by $[c_1|\cdots|c_k]$. The twisting morphism ω is the composite

$$C \longrightarrow \bar{C} \xrightarrow{\blacktriangle} \blacktriangledown\bar{C} \longrightarrow T(\blacktriangledown\bar{C}) = \Omega(C),$$

explicitly, $\omega(c) = [c]$. The differential on $\Omega(C)$ is then specifically crafted for ω to indeed be a twisting morphism:

$$\begin{aligned}\partial_k = &\sum_{i=1}^{k-1} \mathrm{id}^{\otimes i-1} \otimes D \otimes \mathrm{id}^{\otimes k-i-1} \\ &+ \sum_{i=1}^{k} \mathrm{id}^{\otimes i-1} \otimes \partial \otimes \mathrm{id}^{\otimes k-i}\end{aligned}$$

where D is the comultiplication of C.

Theorem 4.1. *(1) The twisted complexes* $\mathrm{B}(A) \otimes_\beta A$ *and* $C \otimes_\omega \Omega(C)$ *are acyclic (see next remark).*

(2) We have an adjunction

$$dg\mathcal{A}\mathrm{lg}(\Omega(C), A) \cong \mathrm{Tw}(C, A) \cong dgco\mathcal{A}\mathrm{lg}(C, \mathrm{B}(A))$$

and the unit and counit are natural quasi-isomorphisms.

(3) For G *a simplicial group, there is a quasi-isomorphism* $\mathbb{Z}\mathfrak{W}(G) \longrightarrow \mathrm{B}\mathbb{Z}G$. *For* X *a simply connected reduced simplicial set, there is a quasi-isomorphism* $\Omega\mathbb{Z}X \longrightarrow \mathbb{Z}\mathfrak{G}(X)$.

Recall that by convention, acyclicity of $\mathrm{B}(A) \otimes_\beta A$ means that the homology is trivial in every dimension, except in dimension 0 where it is $\Bbbk$. Consequently the bar construction gives a free resolution of the trivial A-module $\Bbbk$. In practice however, it is too big to be computationally tractable...

5. Where it all come together

5.1. The Eilenberg–Zilber theorem. For X a simplicial set, let $\mathbb{Z}X$ be its Moore complex with integer coefficients.

Theorem 5.1 (Eilenberg–Zilber)**.** *Let* X *and* Y *be simplicial sets. There is a homotopy equivalence*

$$\mathbb{Z}[X \times Y] \longrightarrow \mathbb{Z}X \otimes \mathbb{Z}Y.$$

called the Alexander–Whitney *map.*

This result gives rise to a few construction. First, consider the following

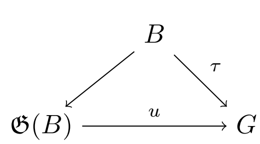
$$\mathbb{Z}X \xrightarrow{\mathbb{Z}[\Delta]} \mathbb{Z}[X \times X] \longrightarrow \mathbb{Z}X \otimes \mathbb{Z}X.$$

This gives a structure of dg-coalgebra on $\mathbb{Z}X$. Let $G \circlearrowleft X$ be an action of a simplicial group G on X. Then we can lift this action into the *Pontryagin product*

$$\mathbb{Z}G \otimes \mathbb{Z}X \longrightarrow \mathbb{Z}[G \times X] \longrightarrow \mathbb{Z}X.$$

For $X = G$, this construction gives a dg-algebra structure on $\mathbb{Z}G$ (compatible with the coalgebra structure above, producing in fact a Hopf algebra?), and in general, a left $\mathbb{Z}G$-dg-module structure on $\mathbb{Z}X$.

Let $G \longrightarrow E \xrightarrow{p} B$ be a principal bundle, σ a pseudo section of p, and $\tau : B \longrightarrow G$ the associated twisting map. Thus, $E \cong B \times_\tau G$. Further, there is a universal morphism



that induces a twisting map t defined as the composite

$$\mathbb{Z}B \xrightarrow{f} \mathbb{Z}\mathfrak{G}(B) \xrightarrow{\mathbb{Z}u} \mathbb{Z}G,$$

where the twisting map f is constructed using acyclic models.

Theorem 5.2 (Twisted Eilenberg–Zilber, due to Brown)**.** *There is a natural homotopy equivalence*

$$\mathbb{Z}\left[B \times_\tau G\right] \longrightarrow \mathbb{Z}B \otimes_t \mathbb{Z}G.$$

6. Koszul algebras

Let R be a ring. For $M \in {}_R\mathcal{M}\mathrm{od}$, recall that $\mathrm{Ext}^\bullet(-, M)$ is the left derived functor of $\hom(-, M)$. Specifically, for $N \in {}_R\mathcal{M}\mathrm{od}$, choose a projective resolution $P^{(\bullet)} \to N \to 0$, and define $\mathrm{Ext}^\bullet(N, M)$ as the cohomology $H^\bullet(\hom(P^{(\bullet)}, M))$.

Let now R be a $\Bbbk$-algebra. Koszul duality takes interest in providing a simple resolution of $\Bbbk$ seen as a trivial R-module.

Lemma 6.1. *Let* $M = \bigoplus_i M_i$ *be a graded* R*-module living only in degree* $\geq n$ *(i.e.* $M_{<n} = 0$*), for a fixed* n*. Then* M *admit a free resolution* $P^{(\bullet)} \to M \to 0$ *such that* $P^{(i)}$ *only lives in degree* $\geq n + i$.

Lemma 6.2. *Let* $0 \to K \to P^{(n-1)} \to \cdots \to P^{(0)} \to \Bbbk \to 0$ *be an exact sequence, where* $P^{(i)}$ *is projective, and where* K *only lives in degree* $\geq l$ *for some* $l > 0$ *(take for instance a partial projective resolution as in the previous lemma). Then* $\mathrm{Ext}^n(\Bbbk, N) \cong \hom(K, N)$ *for any* N.

The algebra R is said *Koszul* if it is reduced ($R_0 = \Bbbk$), and if either of the following equivalent conditions hold:

(1) the trivial R-module $\Bbbk$ has a projective resolution $P^{(\bullet)} \to \Bbbk \to 0$ such that $P^{(i)}$ is generated in degree i (i.e. $P^{(i)} = RP^{(i)}_i$);
(2) we have $\mathrm{Ext}^i(\Bbbk, \blacktriangle^j\Bbbk) = 0$ if $i \neq j$.

Proposition 6.3. *If* A *is Koszul, then it is quadratic (see next section). Indeed,* $\mathrm{Ext}^\bullet(\Bbbk, \blacktriangle^2\Bbbk)$ *has to be concentrated in dimension* 2*, and some resolution shenanigans show that* A *is quadratic* [Wu16, Proposition 4.2.11]

7. Koszul duality

Let V be a graded $\Bbbk$-vector space. The *tensor algebra* on V is defined as

$$T(V) = \bigoplus_{i \in \mathbb{N}} V^{\otimes i}$$

with the concatenation product. Although homogeneous elements of $T(V)$ are of the form $v = v_1 \otimes \cdots \otimes v_k$, we omit the $\otimes$ by writing $v = v_1 \cdots v_k$. Note a homogeneous elements $v = v_1 v_2 \cdots v_n$ of $T(V)$ have both a degree $|v| = \sum_i |v_i|$ and a weight $w(v) = n$.

Let Q be a graded subspace of $V^{\otimes 2}$. The *quadratic algebra* $A(V, Q)$ of the quadratic data [LV12] (V, Q) is defined as

$$A(V, Q) = T(V) / \langle Q \rangle,$$

where specifically, $\langle Q \rangle_0 = \langle Q \rangle_1 = 0$, and $\langle Q \rangle_i = \sum_{j=0}^{i-2} V^{\otimes j} \otimes Q \otimes V^{\otimes i-j-2}$. The multiplication is the usual concatenation operation on the tensor algebra $T(V)$ [LV12], and the algebra is endowed with trivial differential. The *quadratic coalgebra* of (V, Q) is

$$C(V, Q) = \Bbbk \oplus V \oplus Q \oplus \bigoplus_{i > 2} \bigcap_{j=0}^{i-2} V^{\otimes j} \otimes Q \otimes V^{\otimes i-j-2}.$$

The comultiplication is the usual deconcatenation operation on $T^c(V)$, and the coalgebra is endowed with trivial differential.

The *Koszul dual algebra* of the quadratic algebra $A = A(V, Q)$ is defined as

$$A^! = A(V^*, Q^\perp),$$

endowed with trivial differential, where $Q_i^\perp = \{\alpha \in (V^*)^{\otimes i} \mid \alpha(q) = 0, \forall q \in \langle Q\rangle_i\}$. The *Koszul dual coalgebra* is defined as

$$A^{¡} = C(\blacktriangle V, \blacktriangle^2 Q)$$

endowed with trivial differential. Note that there is a twisting map $\kappa : A^{¡} \longrightarrow A$ defined as the composite

$$A^{¡} = C(\blacktriangle V, \blacktriangle^2 Q) \longrightarrow \blacktriangle V \xrightarrow{\blacktriangledown} V \longrightarrow A(V, R) = A,$$

which gives rise to the twisted complexes $A^{¡} \otimes_\kappa A$ and $A \otimes_\kappa A^{¡}$, called left and right *Koszul complex* respectively. Expliciently, the differential on $A^{¡} \otimes_\kappa A$ is

$$\partial_\kappa (q_1 \cdots q_{k-1} q_k \otimes a) = q_1 \cdots q_{k-1} \otimes q_k a,$$

and similarly for $A \otimes_\kappa A^{¡}$.

Theorem 7.1. *Let A be a quadratic algebra. The following are equivalent:*

(1) the algebra A is Koszul;
(2) the dual algebra $A^!$ is Koszul;
(3) the left complex $A^{¡} \otimes_\kappa A$ is acyclic (see next remark);
(4) the right complex $A \otimes_\kappa A^{¡}$ is acyclic (see next remark).

Since this complex $A \otimes_\kappa A^{¡}$ is augmented, "acyclic" means by convention that the homology is 0 in every dimension except 0 where it is $\Bbbk$. Observe that the weight n component of $A^{¡} \otimes_\kappa A$ reads

$$0 \to A^{¡(n)} \to A^{¡(n-1)} \otimes A^{(1)} \to \cdots \to A^{¡(1)} \otimes A^{(n-1)} \to A^{(n)} \to 0$$

thus summing them all up and appending $\Bbbk$ give an augmented complex

$$\cdots \to A^{¡(n)} \otimes A \to \cdots \to A^{¡(1)} \otimes A \to A \xrightarrow{\varepsilon} \Bbbk \to 0$$

which, if A is Koszul, is a resolution of $\Bbbk$ by free A-modules. We can proceed similarly with $A \otimes_\kappa A^{¡}$. Note that in the left Koszul complex, the n-th term $A^{¡(n)} \otimes A$ is generated in weight (n), and this provide a resolution of $\Bbbk$ as in the definition of a Koszul algebra above. Thus a method for proving koszulity is to show that the left (or right) Koszul complex is exact.

8. Examples

Example 8.1. Let V be finite dimensional. The dual of the tensor algebra $T(V)$ is $T(V)^! = \Bbbk \oplus V^*$ with trivial multiplication, called the *algebra of dual numbers*. Indeed, $T(V) = T(V)/\langle Q\rangle$ for $Q = \{0\}$, thus $Q^\perp = \bigoplus_{i\geq 2}(V^*)^{\otimes i}$. Moreover, the dual coalgebra is $T(V)^{¡} = \Bbbk \oplus V$, whence the augmented right Koszul complex is

$$0 \to T(V) \otimes V \xrightarrow{\partial_0} T(V) \xrightarrow{\varepsilon} \Bbbk \to 0,$$

where $\partial(\overrightarrow{v_i} \otimes w) = \overrightarrow{v_i} w$, and ε is the augmentation. One readily checks that the complex is acyclic (meaning exact in this context), whence $T(V)$ and $T(V)^!$ are Koszul.

Example 8.2. Let V be finite dimensional. The *symmetric algebra* over V is given by $S(V) = T(V)/\langle Q\rangle$ where $Q = \{vw - wv \mid v, w \in V\}$. The dual of $S(V)$ is the *exterior algebra* $\bigwedge(V) = T(V)/\langle R\rangle$ where $R = \{vv \mid v \in V\}$. In [Wu16, Proposition 4.1.8], it is proved that both are Koszul.

See [LV12, Wu16] for more examples.

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