STABILITY OF LAWVERE THEORIES

CÉDRIC HO THANH

Abstract. TODO: Write the abstract.

Contents

1. Introduction

TODO: Write the introduction

1.1. **Prerequisites.** Let $\mathcal{C}$ be a category with products. Let I be some index set, and $d_i \in \mathcal{C}$, for $i \in I$. The projection onto the i-th component is denoted by $\pi_i : \prod_j d_j \longrightarrow d_i$. If all the d_i's are equal to some $d \in \mathcal{C}$, we write $d^I \coloneqq \prod_{i \in I} d_i$. By universal property, a collection of morphisms $g_i : c \longrightarrow d_i$ induces a morphism

$$\langle g_i \rangle_{i \in I} : c \longrightarrow \prod_{i \in I} d_i. \tag{1.1}$$

If all the g_i's are identities, then $\Delta_I \coloneqq \langle \mathrm{id}_c \rangle_{i \in I} : c \longrightarrow c^I$ is called a *diagonal* of c. We abbreviate Δ_I as simply Δ if no ambiguity arise. If $f : b \longrightarrow c$ and $h_i : d_i \longrightarrow e_i$, then

$$\langle h_i g_i f \rangle_i = \left(\prod_i h_i \right) \langle g_i \rangle_i f. \tag{1.2}$$

Date: 2021 .

In particular, $\langle g_i \rangle_i = (\prod_i g_i) \Delta$. A bijection $\sigma : I \longrightarrow I$ induces an automorphism

$$\sigma := \left\langle \pi_{\sigma(i)} \right\rangle_{i \in I} : d^I \longrightarrow d^I. \tag{1.3}$$

We denote by S_n the n-th symmetric group, which is the group of automorphisms of an n-element set.

2. Lawvere theories

2.1. **Definition.** For $n \in \mathbb{N}$, let $[n]$ be the n-elements set $\{1, 2, \ldots, n\}$. If no ambiguity arises, we shall simply denote it by $n = [n]$. Let $\aleph_0$ be the full subcategory of $\mathcal{S}\mathrm{et}$ spanned by sets of the form n. It is a skeleton of $\mathcal{S}\mathrm{et}_f$, the category of finite sets.

Clearly, $\aleph_0$ has finite coproducts, given by addition of numbers. Therefore, $\aleph_0^{\mathrm{op}}$ has finite products, also given by addition of numbers. In particular, all objects of $\aleph_0^{\mathrm{op}}$ are iterated products of the object 1.

A *Lawvere theory* (also called *one-storted algebraic theory* [ARV11, definition 11.3]) is a category L with finite products, equipped with a functor $l : \aleph_0^{\mathrm{op}} \longrightarrow \mathsf{L}$ that is an identity on objects and preserves finite products. In other words, L is a category with finite products, where all objects are iterated products of a distinguished object 1. Therefore, the functor l shall remain implicit. A morphism $F : \mathsf{L} \longrightarrow \mathsf{K}$ between Lawvere theories is simply a functor below $\aleph_0^{\mathrm{op}}$. Let $\mathcal{L}\mathrm{aw}$ be the category of Lawvere theories and such morphisms. It has an initial object, namely $\aleph_0^{\mathrm{op}}$ itself.

2.2. **Models.** Let $\mathcal{C}$ be a category with finite products. A *model* of L in $\mathcal{C}$ is a finite product preserving functor $X : \mathsf{L} \longrightarrow \mathcal{C}$. If no ambiguity arise, we write X^n instead of Xn, X instead of $X^1 = X1$, and if $f \in \mathcal{L}(m, n)$, we write $f : X^m \longrightarrow X^n$ instead of Xf. With these notations, a model X is the datum of an object $X \in \mathcal{C}$ together with a morphism $f : X^n \longrightarrow X$ for every morphism $f \in \mathsf{L}(n, 1)$, satisfying the same relations as L. A morphism of models $\alpha : X \longrightarrow Y$ is simply a natural transformation. Let $\mathsf{L}(\mathcal{C})$ be the category of models of L in $\mathcal{C}$ and $U_{\mathsf{L}} : \mathsf{L}(\mathcal{C}) \longrightarrow \mathcal{C}$ be the forgetful functor. It maps a model X to $X = X^1$, and $\alpha : X \longrightarrow Y$ to $\alpha = \alpha_1$.

Lemma 2.1. *If $\alpha : X \longrightarrow Y$ is a morphism of models, then $\alpha_n : X^n \longrightarrow Y^n$ is simply $(\alpha_1)^n$.*

Proof. Consider the projection $\pi_i : n \longrightarrow 1$ in L. Since α is a natural transformation, the following square commutes:

$$\begin{tikzcd} X^n \arrow[r, "\alpha_n"] \arrow[d, "\pi_i"'] & Y^n \arrow[d, "\pi_i"] \\ X \arrow[r, "\alpha_1"] & Y. \end{tikzcd} \tag{2.2}$$

Since this holds for all $1 \leq i \leq n$, we conclude that $\alpha_n = (\alpha_1)^n$. □

Lemma 2.3. *Let $X, Y \in \mathsf{L}(\mathcal{C})$ and $\beta : X \longrightarrow Y$ be a morphism in $\mathcal{C}$. Then β extends as a morphism of models $X \longrightarrow Y$ if and only if for all $n \in \mathbb{N}$ and $f \in \mathsf{L}(n, 1)$, the* *following square commutes*

$$\begin{tikzcd} X^n \arrow[r, "\beta^n"] \arrow[d, "f"'] & Y^n \arrow[d, "f"] \\ X \arrow[r, "\beta"] & Y. \end{tikzcd} \tag{2.4}$$

Proof. Necessity is clear. We now prove that the condition is sufficient. Note that a morphism $g : n \longrightarrow m$ in L can be decomposed as $g = \langle \pi_i g \rangle_{1 \le i \le m}$, and that each $\pi_i g$ is a morphism $n \longrightarrow 1$. By assumption, $\pi_i g \beta^n = \beta \pi_i g$, thus

$$g\beta^n = \langle \pi_i g \beta^n \rangle_{1 \le i \le m} = \langle \beta \pi_i g \rangle_{1 \le i \le m} = \beta g. \tag{2.5}$$

□

Thanks to lemmas 2.1 and 2.3, we can write the n-th component of $\alpha : X \longrightarrow Y$ simply as $\alpha^n : X^n \longrightarrow Y^n$. Consider the case $\mathcal{C} = \mathcal{S}\mathrm{et}$, and let $n \in \mathbb{N}$. The *n-th representable model of* L [ARV11, remark 1.12] is $L_n \coloneqq \mathsf{L}(n, -) : \mathsf{L} \longrightarrow \mathcal{S}\mathrm{et}$.

Lemma 2.6 (Yoneda lemma)**.** *For $X \in \mathsf{L}(\mathcal{S}\mathrm{et})$ there is an natural isomorphism (of sets)*

$$X^n \cong \mathsf{L}(\mathcal{S}\mathrm{et})(L_n, X). \tag{2.7}$$

More precisely, it maps an element $x \in X^n$ to the unique morphism $\tilde{x} : L_n \longrightarrow X$ such that $\tilde{x}^n(\mathrm{id}_n) = x$.

2.3. **Products of models.** By [AR94, theorem 1.46 and corollary 1.52], $\mathsf{L}(\mathcal{S}\mathrm{et})$ is a finitely locally presentable category, and in particular, it has all limits and colimits. The same result cannot be expected to hold for a general ground category $\mathcal{C}$ with finite products, but as we will see in lemma 2.11, $\mathsf{L}(\mathcal{C})$ still has finite products.

For integers $m, n > 0$, let $\tau_{n;m} \in S_{mn}$ be the *shuffle permutation*, i.e. the permutation such that for all $1 \le i \le m$ and $1 \le j \le n$, we have

$$\tau_{n;m}((j-1)n + i) = (i-1)m + j. \tag{2.8}$$

In other words, $\tau_{n;m}$ "rearranges m tuples of n elements into n tuples of m elements". If n or m is 0, then by convention, let $\tau_{n;m}$ be the identity on the empty set. We consider $\tau_{n;m}$ as a morphism $mn \longrightarrow mn$ of $\aleph_0^{\mathrm{op}}$, and therefore as a morphism in any Lawvere theory.

If $c_1, \ldots, c_n \in \mathcal{C}$ and $m \in \mathbb{N}$, then there is an isomorphism

$$\tau = \tau_{c_1,\ldots,c_n;m} : \left(\prod_i c_i\right)^m \longrightarrow \prod_i c_i^m \tag{2.9}$$

induced by $\tau_{n;m}$, which again, rearranges m tuples of n elements into n tuples of m elements. Explicitly, if $p_k : \left(\prod_i c_i\right)^m \longrightarrow c_l$ is the projection into the k-th component, where $1 \le k \le mn$, $1 \le l \le n$, and $l \equiv k \mod n$, then τ is the universal morphism

$$\tau = \left\langle p_{\tau_{n;m}(k)} \right\rangle_{1 \le k \le mn}. \tag{2.10}$$

If all the c_i's are equal to some object c, then we write $\tau_{c;n;m}$ instead of $\tau_{c,\ldots,c;m}$.

Lemma 2.11 ([ARV11, proposition 1.21])**.** *The category $\mathsf{L}(\mathcal{C})$ has finite products, given as follows:*

(1) if the terminal object of $\mathcal{C}$ is 1, then the terminal model is the constant functor at 1;

(2) if $X, Y \in \mathsf{L}(\mathcal{C})$, then $X \times Y$ is the functor

$$\begin{aligned} X \times Y : \mathsf{L} &\longrightarrow \mathcal{C} \\ n &\longmapsto (X \times Y)^n && n \in \mathbb{N} \\ f &\longmapsto \tau_{X,Y;q}^{-1}(f \times f)\tau_{X,Y;p} && f \in \mathsf{L}(p,q). \end{aligned}$$

In details, $(X \times Y)f$ is the composite

$$(X \times Y)^p \xrightarrow{\tau_{X,Y;p}} X^p \times Y^p \xrightarrow{f \times f} X^q \times Y^q \xrightarrow{\tau_{X,Y;q}^{-1}} (X \times Y)^q. \tag{2.12}$$

Proof. The model described in point (1) is clearly terminal. Consider $X \times Y$ as defined in point (2), and define $\pi_{X,n}$ as the composite

$$(X \times Y)^n \xrightarrow{\tau_{X,Y;n}} X^n \times Y^n \xrightarrow{\pi_{X^n}} X^n. \tag{2.13}$$

Take $f \in \mathsf{L}(p,q)$, and consider

$$\begin{tikzcd} (X \times Y)^p \arrow[r, "\tau_{X,Y;p}"] \arrow[d, "(X \times Y)f"] \arrow[rr, bend left, "\pi_{X,p}"] & X^p \times Y^p \arrow[r, "\pi_{X^p}"] \arrow[d, "f \times f"] & X^p \arrow[d, "f"] \\ (X \times Y)^q \arrow[r, "\tau_{X,Y;q}"] \arrow[rr, bend right, "\pi_{X,q}"'] & X^q \times Y^q \arrow[r, "\pi_{X^q}"] & X^q. \end{tikzcd} \tag{2.14}$$

Both inner squares commute by definition, thus so does the outer one. In other words, $\pi_{X,p}$'s jointly define a natural transformation $\pi_X : X \times Y \longrightarrow X$. Similarly, we define $\pi_Y : X \times Y \longrightarrow Y$. We now check that $X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y}$ is a limit cone. For $X \xleftarrow{\alpha} Z \xrightarrow{\beta} Y$ another cone, let γ_n be the following composite

$$Z^n \xrightarrow{\langle \alpha_n, \beta_n \rangle} X^n \times Y^n \xrightarrow{\tau_{X,Y;n}^{-1}} (X \times Y)^n. \tag{2.15}$$

Akin to (2.14), it is easy to check that the γ_n's jointly define a morphism $Z \longrightarrow X \times Y$, and that furthermore, $\alpha = \pi_X \gamma$ and $\beta = \pi_Y \gamma$. If γ' is another such morphism, then necessarily

$$\tau_{X,Y;2}\gamma'_n = \langle \alpha_n, \beta_n \rangle = \tau_{X,Y;2}\gamma, \tag{2.16}$$

and since $\tau_{X,Y;2}$ is an isomorphism, $\gamma'_n = \gamma_n$. Finally, $X \times Y$ defined above is the product of X and Y. $\square$

Lemma 2.17. *If $X \in \mathsf{L}(\mathcal{C})$, $g \in \mathsf{L}(n,1)$, and $m \in \mathbb{N}$, then*

$$X^m g = (Xg)^m \tau_{X;m;n} = X\left(g^m \tau_{m;n}\right). \tag{2.18}$$

Proof. The second equality follows from the fact that X preserves finite products. For the first one, we proceed by induction. The cases $m = 0, 1$ hold trivially, and $m = 2$ holds by definition of the product in $\mathsf{L}(\mathcal{C})$. Thus, assume that $m \geq 3$. We have

$$
\begin{aligned}
X^m g &= (X \times X^{m-1}) g \\
&= \tau^{-1}_{X, X^{m-1}; 1} (Xg \times X^{m-1} g) \tau_{X, X^{m-1}; 1} \\
&= (Xg \times X^{m-1} g) \tau_{X, X^{m-1}; 1} && \tau_{X, X^{m-1}; 1} = \mathrm{id} \\
&= (Xg \times (Xg)^{m-1} \tau_{X; m-1; n}) \tau_{X, X^{m-1}; 1} && \text{by induction} \\
&= (Xg)^m (\mathrm{id}_X \times \tau_{X; m-1; n}) \tau_{X, X^{m-1}; 1} \\
&= (Xg)^m \tau_{X; m; n}.
\end{aligned}
$$

□

3. Tensor product

3.1. **Commutativity.** Let $\mathsf{K}, \mathsf{L} \in \mathcal{L}\mathrm{aw}$, and $\mathcal{C}$ be a category with finite products. As we saw in lemma 2.11, $\mathsf{L}(\mathcal{C})$ has finite products, and therefore, we may consider the category of K-models in $\mathsf{L}(\mathcal{C})$, i.e. $\mathsf{K}(\mathsf{L}(\mathcal{C}))$. As we will see in this section, the latter can be seen as the category of models in $\mathcal{C}$.

Let $f \in \mathsf{L}(n, 1)$ and $g \in \mathsf{L}(m, 1)$. We say that f *commutes* with g, denoted by $f \boxtimes g$, if the following equality is satisfied:

$$f g^n = g f^m \tau_{m;n}. \tag{3.1}$$

More generally, if $c \in \mathcal{C}$, $f \in \mathcal{C}(c^n, 1)$ and $g \in \mathcal{C}(c^m, 1)$, then we say that f commutes with g if $f g^n = g f^m \tau_{c;m;n}$, i.e. if the following diagram commutes:

$$
\begin{tikzcd}
c^{mn} \arrow[r, "g^n"] \arrow[d, "\tau_{c;m;n}"'] & c^n \arrow[r, "f"] & c \arrow[d, equal] \\
c^{nm} \arrow[r, "f^m"] & c^m \arrow[r, "g"] & c.
\end{tikzcd}
\tag{3.2}
$$

Note that since $\tau^{-1}_{m;n} = \tau_{n;m}$, the commutativity relation $\boxtimes$ is symmetric.

To simplify notations, if $A, B \subseteq \mathsf{L}/1$ are sets of morphisms with codomain 1, then we write $A \boxtimes B$ to signify that every morphism in A commutes with every morphism of B. Let $\mathsf{L}|_A$ be the smallest subtheory of L containing A. In particular, $\mathsf{L}|_A/1$ is the smallest subset of $\mathsf{L}/1$ such that

(1) $A \subseteq \mathsf{L}|_A/1$;
(2) the projection $\pi_i : n \longrightarrow 1$ is in $\mathsf{L}|_A/1$, for all $n \in \mathbb{N}$ and $1 \leq i \leq n$;
(3) if $g, f_1, \ldots, f_n \in \mathsf{L}|_A/1$, then $g \prod_i f_i \in \mathsf{L}|_A/1$ and $g \langle f_i \rangle \in \mathsf{L}|_A/1$.

If $\mathsf{L} = \mathsf{L}|_A$, then we say that A *generates* L. Trivially $\mathsf{L}/1$ generates L.

Lemma 3.3. *Let $A, B \subseteq \mathsf{L}/1$. If $A \boxtimes B$, then $(\mathsf{L}|_A/1) \boxtimes (\mathsf{L}|_B/1)$.*

Proof. By symmetry of $\boxtimes$, it is enough to show that $A \boxtimes (\mathsf{L}|_B/1)$. The rest is routine verifications. □

3.2. **Tensor product.** For $\mathsf{L}, \mathsf{K} \in \mathcal{L}\mathrm{aw}$, consider the theory $\mathsf{K} \coprod_{\aleph_0^{\mathrm{op}}} \mathsf{L}$ given by the categorical pushout

$$
\begin{tikzcd}
\aleph_0^{\mathrm{op}} \arrow[r, "k"] \arrow[d, "l"'] & \mathsf{K} \arrow[d] \\
\mathsf{L} \arrow[r] & \mathsf{K} \coprod_{\aleph_0^{\mathrm{op}}} \mathsf{L}. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\tag{3.4}
$$

It contains all the morphisms of K and L, but is quotiented in such a way that only one copy of $\aleph_0^{\mathrm{op}}$ is present. In particular, $\mathsf{K} \coprod_{\aleph_0^{\mathrm{op}}} \mathsf{L}$ is still a Lawvere theory. Let $\mathsf{K} \otimes \mathsf{L}$, the *tensor product* (also called *Kronecker product*) of K and L [Fre66] be the following quotient:

check

$$\mathsf{K} \otimes \mathsf{L} := \frac{\mathsf{K} \coprod_{\aleph_0^{\mathrm{op}}} \mathsf{L}}{fg^n \sim gf^m \tau_{m;n}, \;\; f \in \mathsf{K}(n,1), g \in \mathsf{L}(m,1)}. \tag{3.5}$$

Denote by $i_{\mathsf{K}} : \mathsf{K} \longrightarrow \mathsf{K} \otimes \mathsf{L}$ the natural map (which is not necessarily faithful!), and likewise for L. Since $\tau_{m;n}^{-1} = \tau_{n;m}$, we have $fg^n = gf^m \tau_{m;n}$ if and only if $gf^m \sim fg^n \tau_{n;m}$, which implies that the tensor product $\otimes$ is commutative.

Proposition 3.6. *The initial theory $\aleph_0^{\mathrm{op}}$ is a neutral element for the tensor product.*

Proof. Take $\mathsf{K} \in \mathcal{L}\mathrm{aw}$. Then $\mathsf{K} \coprod_{\aleph_0^{\mathrm{op}}} \aleph_0^{\mathrm{op}} = \mathsf{K}$, thus

$$\begin{aligned} \mathsf{K} \otimes \aleph_0^{\mathrm{op}} &= \frac{\mathsf{K}}{fg^n \sim gf^m \tau_{m;n}, \;\; f \in \mathsf{K}(n,1), g \in \aleph_0^{\mathrm{op}}(m,1)} \\ &= \frac{\mathsf{K}}{f\pi_i^n \sim \pi_i f^m \tau_{m;n}, \;\; f \in \mathsf{K}(n,1), \pi_i : m \longrightarrow 1, 1 \le i \le m}. \end{aligned}$$

However, the relation $f\pi_i^n = \pi_i f^m \tau_{m;n}$ is already satisfied in K (and indeed, in any Lawvere theory), thus $\mathsf{K} \otimes \aleph_0^{\mathrm{op}} = \aleph_0^{\mathrm{op}} \otimes \mathsf{K} = \mathsf{K}$. ☐

Lemma 3.7. *Let $X \in \mathsf{K}(\mathsf{L}(\mathcal{C}))$.*

(1) *For $d \in \hom \aleph_0^{\mathrm{op}}$ we have $(X1)d = (Xd)_1$.*
(2) *For $f \in \mathsf{K}/1$ and $g \in \mathsf{L}/1$ we have $(X1)g \boxtimes (Xf)_1$.*

Proof. (1) Since X and $X1$ are models of Lawvere theories, they preserve finite products.
(2) By naturality of Xf, the following diagram commutes

$$\begin{tikzcd} (Xm)n \arrow[r, "(Xf)_n"] \arrow[d, "(Xm)g"'] & (X1)n \arrow[d, "(X1)g"] \\ (Xm)1 \arrow[r, "(Xf)_1"] & (X1)1. \end{tikzcd} \tag{3.8}$$

By lemma 2.11, $(Xm)g = (X1)^m g = ((X1)g)^m \tau_{X1;m;n}$, thus

$$(Xf)_1((X1)g)^m \tau_{(X1)1;m;n} = (X1)g(Xf)_n = ((X1)g)(Xf)_1^n, \tag{3.9}$$

which is the desired relation. ☐

Theorem 3.10. *There is an equivalence $\mathsf{K}(\mathsf{L}(\mathcal{C})) \simeq \mathsf{K} \otimes \mathsf{L}(\mathcal{C})$.*

Proof. Informally, the central observation is that the relations $fg^n = gf^m \tau_{m;n}$ ranging over g encodes the fact that f induces a natural transformation between L-models in $\mathcal{C}$, i.e. a morphism in $\mathsf{L}(\mathcal{C})$.

Let $X \in \mathsf{K}(\mathsf{L}(\mathcal{C}))$. By lemma 3.7, the following functor is a well-defined model of $\mathsf{K} \otimes \mathsf{L}$:

$$\begin{aligned} X^\flat : \mathsf{K} \otimes \mathsf{L} &\longrightarrow \mathcal{C} \\ n &\longmapsto (X1)n && n \in \mathbb{N} \\ f &\longmapsto (Xf)_1 && f \in \hom \mathsf{K} \\ g &\longmapsto (X1)g && g \in \hom \mathsf{L}. \end{aligned}$$

This defines a functor $(-)^\flat : \mathsf{K}(\mathsf{L}(\mathcal{C})) \longrightarrow \mathsf{K} \otimes \mathsf{L}(\mathcal{C})$ mapping a morphism α to $\alpha^\flat := \alpha_1$. Conversely, let $Y \in \mathsf{K} \otimes \mathsf{L}(\mathcal{C})$. For $f \in \mathsf{L}(m, m')$ and $g \in \mathsf{L}(n, 1)$, the following square commutes

$$\begin{tikzcd} Y^{mn} \arrow[r, "(Yf)^n"] \arrow[d, "Y^m g"'] & Y^{m'n} \arrow[d, "Y^{m'} g"] \\ Y^m \arrow[r, "Yf"] & Y^{m'} \end{tikzcd} \tag{3.11}$$

since in $\mathsf{K} \otimes \mathsf{L}$ we have $\pi_i f \boxtimes g$ for all $1 \le i \le m$. Therefore, by lemma 2.3, the $(Yf)^n$'s jointly define a morphism $(Yf)^\bullet : Y^m i_{\mathsf{K}} \longrightarrow Y^{m'} i_{\mathsf{K}}$. This enables the following definition

$$\begin{aligned} Y^\sharp : \mathsf{K} &\longrightarrow \mathsf{L}(\mathcal{C}) & \\ n &\longmapsto Y^n i_{\mathsf{K}} & n \in \mathbb{N} \\ f &\longmapsto (Yf)^\bullet & f \in \hom \mathsf{K}, \end{aligned}$$

from which we derive a functor $(-)^\sharp : \mathsf{K} \otimes \mathsf{L}(\mathcal{C}) \longrightarrow \mathsf{K}(\mathsf{L}(\mathcal{C}))$ mapping a morphism β to $(\beta_1)^\bullet$. Finally, it is routine verification to show that $(-)^\flat$ and $(-)^\sharp$ are mutually inverse equivalences of categories. □

Corollary 3.12. *We have* $\mathsf{K}(\mathsf{L}(\mathcal{C})) \simeq \mathsf{L}(\mathsf{K}(\mathcal{C}))$.

3.3. Derived theories.

find better term for that

Let Mon be the theory of monoids. It is generated by a constant $0 : 0 \longrightarrow 1$, a binary operation $\lambda = \lambda_2 : 2 \longrightarrow 1$, and the following relations:

$$\lambda(\mathrm{id}_1 \times 0) = \mathrm{id}_1 = \lambda(0 \times \mathrm{id}_1), \qquad \lambda(\lambda \times \mathrm{id}_1) = \lambda(\mathrm{id}_1 \times \lambda), \tag{3.13}$$

standing for unitality and associativity, respectively. Define the n-ary multiplication λ_n as follows: $\lambda_0 := 0$, $\lambda_1 := \mathrm{id}_1$, and for $n \ge 2$, $\lambda_{n+1} := \lambda(\lambda_n \times \mathrm{id}_1)$. Let cMon be the theory of commutative monoids, generated like Mon but with the following additional commutativity axiom:

$$\lambda = \lambda(1\,2). \tag{3.14}$$

Finally, the theory Ab of abelian groups extends cMon with a morphism $i : 1 \longrightarrow 1$ and the following invertibility axiom

$$\lambda(i \times \mathrm{id}_1) = 0 = \lambda(\mathrm{id}_1 \times i). \tag{3.15}$$

Proposition 3.16 (Eckmann–Hilton argument)**.** *Let* K *be a Lawvere theory,* $n \ge 2$*,* $f, g : n \longrightarrow 1$*, and* $0 : 0 \longrightarrow 1$*. Assume that* 0 *is a neutral element for* f *and* g*, i.e.* $f(0^{i-1} \times \mathrm{id}_1 \times 0^{n-i}) = \mathrm{id}_1$*, for all* $1 \le i \le n$*, and likewise for* g*. If* $f \boxtimes g$*, then* $f = g$ *and for all* $\sigma \in S_n$*,* $f = f\sigma$*.*

Proof. Take $\sigma \in S_n$, and define

$$x_{i,j} := \begin{cases} \mathrm{id}_1 & \text{if } i = \sigma(j) \\ 0 & \text{otherwise.} \end{cases} \tag{3.17}$$

We have

$$\begin{aligned}
f &= f \prod_i x_{i,\sigma(i)} \\
&= f \prod_i g \prod_j x_{i,j} && \text{since 0 neutral for } g \\
&= f g^n \prod_{i,j} x_{i,j} \\
&= g f^n \tau_{n;n} \prod_{i,j} x_{i,j} && \text{since } f \boxtimes g \\
&= g f^n \left(\prod_{i,j} x_{j,i} \right) \tau_{n;n} \\
&= g \left(\prod_i f \prod_j x_{j,i} \right) \tau_{n;n} && \text{since 0 neutral for } f \\
&= g \left(\prod_i x_{\sigma(i),i} \right) \tau_{n;n} \\
&= g\sigma
\end{aligned}$$

since $\tau_{n;n}((i-1)n + \sigma(i)) = (\sigma(i) - 1)n + i$. Taking $\sigma = \mathrm{id}_n$ we find $f = g$. Then, for a general σ, we have $f = g\sigma = f\sigma$. □

Proposition 3.18 (Classical Eckmann–Hilton argument [EH62])**.** *For all $n \geq 2$, we have $\mathsf{cMon} = \mathsf{Mon}^{\otimes n}$. In particular, $\mathsf{cMon} = \mathsf{cMon} \otimes \mathsf{cMon}$.*

Proof. In $\mathsf{Mon} \otimes \mathsf{Mon}$, denote by 0 and λ the operation from the left instance of Mon, while $0'$ and λ' are those from the right instance. Since $0 \boxtimes 0'$, we immediately conclude that $0 = 0'$. Then, 0 is neutral for both λ and λ', and by definition, $\lambda \boxtimes \lambda'$. Thus, by proposition 3.16, $\lambda = \lambda'$ and λ is commutative. In other words, $\mathsf{cMon} = \mathsf{Mon} \otimes \mathsf{Mon}$. Similarly, $\mathsf{cMon} = \mathsf{cMon} \otimes \mathsf{Mon}$. □

Let $\mathsf{K}, \mathsf{L} \in \mathcal{L}\mathrm{aw}$. We say that K is a *simple* L*-theory* if it can be decomposed as $\mathsf{K} = \mathsf{K}' \otimes \mathsf{L}$, for some $\mathsf{K}' \in \mathcal{L}\mathrm{aw}$. It is an L*-theory* if it is the quotient of a simple L-theory.

The terminology isn't great, as it suggests enrichement.

Let K be an Mon-theory, i.e. a quotient of a theory of the form $\mathsf{K}' \otimes \mathsf{Mon}$. The set $\mathsf{K}(1,1)$ has a semiring structure, where multiplication is given by composition, and where addition is defined as

$$x + y \coloneqq \lambda \langle x, y \rangle \tag{3.19}$$

for $x, y : 1 \longrightarrow 1$. We denote this semiring by $\varepsilon_1 \mathsf{K}$. It is easy to check that the n-ary sum of $x_1, \ldots, x_n : 1 \longrightarrow 1$ is

$$\sum_i x_i = \lambda_n \langle x_i \rangle_i \,. \tag{3.20}$$

Proposition 3.21. *Ab-theories are exactly of the form Mod_R, where R is a commutative ring.*

probably not the best place for it

probably similar results for Mon- and cMon-theories.

Proof. If K is an Ab-theory, then it is easy to see that $\mathsf{K} = \mathsf{Mod}_R$, where $R \coloneqq \varepsilon_1 \mathsf{K}$. □

Lemma 3.22. *Let $A, B \subseteq \mathsf{K}/1$ be such that*

(1) $A \cup B$ generates K;

(2) $A \boxtimes B$.

Then K *is a quotient of* $\mathsf{K}|_A \otimes \mathsf{K}|_B$.

Proof. Since $A \boxtimes B$ in K, the natural functors $\mathsf{K}|_A \longrightarrow \mathsf{K}$ and $\mathsf{K}|_B \longrightarrow \mathsf{K}$ extend to a functor $F : \mathsf{K}|_A \otimes \mathsf{K}|_B \longrightarrow \mathsf{K}$. Since $A \cup B$ generates K, F is full. □

Proposition 3.23. *Let* $\mathsf{K}, \mathsf{L} \in \mathcal{L}\mathrm{aw}$*, where* K *is commutative. Then* L *is a* K*-theory if and only if it there exists a morphism* $F : \mathsf{K} \longrightarrow \mathsf{L}$ *such that* $F(\mathsf{K}/1) \boxtimes \mathsf{L}/1$ *.*

Proof.
(1) ($\Longrightarrow$) Assume that L is a K-theory, i.e. obtained as a quotient of a simple K-theory, say $\mathsf{L}' = \mathsf{L}'' \otimes \mathsf{K}$. Let F be the composite

$$\mathsf{K} \longrightarrow \mathsf{L}'' \otimes \mathsf{K} \longrightarrow \mathsf{L}. \tag{3.24}$$

Let $A := \mathsf{K}/1$ and $B := \mathsf{L}''/1$. Then by definition of the tensor product, in L', we have $A \boxtimes B$. Furthermore, by commutativity, in K we have $A \boxtimes A$. Thus, in L', we have $A \boxtimes A \cup B$. Since L' is generated by $A \cup B$, it follows from lemma 3.3 that $A \boxtimes \mathsf{L}'/1$. Finally, $F(\mathsf{K}/1) = F(A) \boxtimes F(\mathsf{L}'/1) = \mathsf{L}/1$.

(2) ($\Longleftarrow$) Note that $A := F(\mathsf{K}/1)$ and $B := \mathsf{L}/1$ satisfy the requirements of lemma 3.22. □

Corollary 3.25. *A theory* K *is a* cMon*-theory if and only if it contains* cMon *and* $\{0, \lambda\} \boxtimes \mathsf{K}/1$.

Proof. By the Eckmann–Hilton argument (proposition 3.18), cMon is commutative. Apply proposition 3.23. □

To be precise, we need to invoke proposition 5.2

Assume that K is a Mon-theory, and let $f \in \mathsf{K}(n, 1)$. For $1 \leq i \leq n$, the *i-th axis* $f^{[i]}$ of f [BV79] is defined as

$$f^{[i]} := f(0^{i-1} \times \mathrm{id}_1 \times 0^{n-i}). \tag{3.26}$$

Proposition 3.27. *We have* $f = \lambda_n \prod_{i=1}^n f^{[i]}$*. Further, the* $f^{[i]}$*'s are unique for this property.*

Proof. We have

$$\begin{aligned}
f &= f \prod_{i=1}^n \lambda_n (0^{i-1} \times \mathrm{id}_1 \times 0^{n-i}) \\
&= f \lambda_n^n \prod_{i=1}^n (0^{i-1} \times \mathrm{id}_1 \times 0^{n-i}) \\
&= \lambda_n f^n \tau_{n;n} \prod_{i=1}^n (0^{i-1} \times \mathrm{id}_1 \times 0^{n-i}) && \text{since } f \boxtimes \lambda_n \\
&= \lambda_n f^n \prod_{i=1}^n (0^{i-1} \times \mathrm{id}_1 \times 0^{n-i}) \\
&= \lambda_n \prod_{i=1}^n f(0^{i-1} \times \mathrm{id}_1 \times 0^{n-i}) \\
&= \lambda_n \prod_{i=1}^n f^{[i]}.
\end{aligned}$$

If $f = \lambda_n \prod_{i=1}^n f_i$, for some $f_1, \ldots, f_n : 1 \longrightarrow 1$, then

$$\begin{aligned} f^{[i]} &= f(0^{i-1} \times \mathrm{id}_1 \times 0^{n-i}) \\ &= \left(\lambda_n \prod_{j=1}^n f_j \right) (0^{i-1} \times \mathrm{id}_1 \times 0^{n-i}) \\ &= \lambda_n \left(\left(\prod_{j=1}^{i-1} f_j 0 \right) \times f_i \times \left(\prod_{j=i+1}^{n} f_j 0 \right) \right) \\ &= \lambda_n (0^{i-1} \times f_i \times 0^{n-i}) && \text{since } f_j \boxtimes 0 \\ &= f_i. \end{aligned}$$

□

3.4. **Matrix calculus.** Let K be a Mon-theory. Let $f = (f_{i,j})_{1 \le i \le m, 1 \le j \le n}$ be a $m \times n$ matrix whose entries are elements of $\mathsf{K}(1,1)$. It induces a morphism $n \longrightarrow m$ (note the order), as

$$\left\langle \lambda_n \prod_{j=1}^n f_{i,j} : n \longrightarrow 1 \right\rangle_{1 \le i \le m} \tag{3.28}$$

Lemma 3.29. *Conversely, a morphism $g : n \longrightarrow m$ decomposes as $(g_{i,j})_{1 \le i \le m, 1 \le j \le n}$, where*

$$g_{i,j} := (\pi_i g)^{[j]}. \tag{3.30}$$

Therefore, there is a bijection $M_{m \times n}(\mathsf{K}(1,1)) \cong \mathsf{K}(n,m)$.

Proof. We have, $g = \langle \pi_i g \rangle_{1 \le i \le m}$, and by proposition 3.27, $\pi_i g = \lambda_n \prod_{j=1}^n (\pi_i g)^{[j]}$. The bijection follows from the fact that a morphism $g : n \longrightarrow m$ is uniquely determined by its projections $\pi_i g$, which are themselves uniquely determined by their axes, see proposition 3.27. □

In particular, if f is a morphism $m \longrightarrow 1$, then its matrix has a single row whose entries are the axes of f.

Lemma 3.31. *If $f = (f_{i,j})_{1 \le i \le m, 1 \le j \le n} : n \longrightarrow m$ and $g = (g_{j,k})_{1 \le j \le n, 1 \le k \le p} : p \longrightarrow n$, then the composite $fg : p \longrightarrow m$ is given by the expected matrix product*

$$(fg)_{i,k} = \sum_j f_{i,j} g_{j,k}. \tag{3.32}$$

Proof. We have

$$\begin{aligned} (fg)_{i,k} &= (\pi_i f g)^{[k]} && \text{by lemma 3.29} \\ &= \pi_i f g (0^{k-1} \times \mathrm{id}_1 \times 0^{p-k}) && \text{by definition} \\ &= \pi_i f \langle \pi_j g \rangle_j (0^{k-1} \times \mathrm{id}_1 \times 0^{p-k}) \\ &= \pi_i f \left\langle \pi_j g (0^{k-1} \times \mathrm{id}_1 \times 0^{p-k}) \right\rangle_j \\ &= \pi_i f \langle g_{j,k} \rangle_j \\ &= \lambda_n \left(\prod_j f_{i,j} \right) \langle g_{j,k} \rangle_j \\ &= \lambda_n \langle f_{i,j} g_{j,k} \rangle_j \\ &= \sum_j f_{i,j} g_{j,k}. \end{aligned}$$

□

Lemma 3.33. *Let $\sigma \in S_n$ be a permutation. The matrix of the induced morphism $\sigma : n \longrightarrow n$ is given by*

$$(\sigma_{i,j})_{1\le i,j\le n} := \begin{cases} \mathrm{id}_1 & \text{if } i = \sigma(j) \\ 0 & \text{otherwise.} \end{cases} \tag{3.34}$$

Proof. For the time being, let $(s_{i,j})_{1\le i,j\le n}$ be the matrix defined above, and s be the morphism $n \longrightarrow n$ it induces. By definition,

$$s = \left\langle \lambda_n \prod_j s_{i,j} \right\rangle_i = \left\langle \pi_{\sigma(i)} \right\rangle_i = \sigma \tag{3.35}$$

□

The matrix of $\mathrm{id}_n \in S_n$ is thus the identity matrix, also denoted by I_n:

$$I_n := \begin{pmatrix} \mathrm{id}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{id}_1 \end{pmatrix}. \tag{3.36}$$

Lemma 3.37. *(1) For $f : m \longrightarrow 1$, $\sigma \in S_m$, and $1 \le i \le m$, we have $(f\sigma)^{[i]} = f^{[\sigma^{-1}(i)]}$.*

(2) For $f : m \longrightarrow 1$, $g : n \longrightarrow 1$, $1 \le i \le m$, and $1 \le j \le n$, we have $(fg^m)^{[(i-1)n+j]} = f^{[i]}g^{[j]}$. In particular, $f \boxtimes g$ if and only if all the axes of f commute (multiplicatively) with all the axes of g.

Proof. (1) We have

$$f\sigma = \begin{pmatrix} f^{[1]} & \cdots & f^{[m]} \end{pmatrix} \sigma = \begin{pmatrix} f^{[\sigma^{-1}(1)]} & \cdots & f^{[\sigma^{-1}(m)]} \end{pmatrix}. \tag{3.38}$$

(2) We have

$$\begin{aligned} fg^m &= \begin{pmatrix} f^{[1]} & \cdots & f^{[m]} \end{pmatrix} \begin{pmatrix} g & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g \end{pmatrix} \\ &= \begin{pmatrix} f^{[1]} & \cdots & f^{[m]} \end{pmatrix} \begin{pmatrix} g^{[1]} & \cdots & g^{[n]} & 0 & \cdots & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & \cdots & 0 & g^{[1]} & \cdots & g^{[n]} \end{pmatrix} \\ &= \begin{pmatrix} f^{[1]}g^{[1]} & f^{[1]}g^{[2]} & \cdots & f^{[1]}g^{[n]} & f^{[2]}g^{[1]} & \cdots & f^{[m]}g^{[n]} \end{pmatrix}. \end{aligned}$$

□

3.5. **Center.** Let $\mathsf{K} \in \mathcal{L}\mathrm{aw}$. Its *center* $Z(\mathsf{K})$ is the subtheory of K generated by those morphisms $f \in \mathsf{K}/1$ such that $f \boxtimes \mathsf{K}/1$. We say that K is *commutative* if $Z(\mathsf{K}) = \mathsf{K}$. In other words $f \in Z(\mathsf{K})(m,n)$ if and only if $f : X^m \longrightarrow X^n$ is a morphism of models, for all $X \in \mathsf{K}(\mathcal{S}\mathrm{et})$. By lemma 3.3, K is commutative if and only if $\mathsf{K}/1 \boxtimes \mathsf{K}/1$.

Lemma 3.39. *Let $\mathsf{K}, \mathsf{L} \in \mathcal{L}\mathrm{aw}$. Then $Z(\mathsf{K}) \otimes Z(\mathsf{L}) \subseteq Z(\mathsf{K} \otimes \mathsf{L})$.*

Proof. Let $f \in Z(\mathsf{K})$. Then by definition, $f \boxtimes \mathsf{K}/1$ in K, thus the same holds in $\mathsf{K} \otimes \mathsf{L}$. Further, $f \boxtimes \mathsf{L}/1$ in $\mathsf{K} \otimes \mathsf{L}$. Finally, $f \in Z(\mathsf{K} \otimes \mathsf{L})$. The same reasoning applies if $f \in Z(\mathsf{L})$. □

Proposition 3.40. *Let* $\mathrm{id} = \mathrm{id}_{\mathsf{K}(\mathcal{S}\mathrm{et})}$ *be the identity functor of* $\mathsf{K}(\mathcal{S}\mathrm{et})$*. There is a bijection* $Z(\mathsf{K})(m,n) \longrightarrow [\mathrm{id}^m, \mathrm{id}^n]$ *which induces an equivalence* $Z(\mathsf{K}) \longrightarrow \mathcal{I}$*, where* $\mathcal{I}$ *is the full subcategory of* $[\mathsf{K}(\mathcal{S}\mathrm{et}), \mathsf{K}(\mathcal{S}\mathrm{et})]$ *spanned by powers of* id.

Proof. Since $Z(\mathsf{K})(m,n) = Z(\mathsf{K})(m,1)^n$ and $[\mathrm{id}^m, \mathrm{id}^n] = [\mathrm{id}^m, \mathrm{id}]^n$, it is enough to prove the claim in the case $n = 1$. Let $f \in Z(\mathsf{K})(m,1)$. For $X \in \mathsf{K}(\mathcal{S}\mathrm{et})$, let $\alpha_{f,X} \coloneqq Xf : X^m \longrightarrow X$. We check that this is a morphism of models using lemma 2.3: for $g : n \longrightarrow 1$

$$\begin{aligned}
\alpha_{f,X} \circ (X^m g) &= (Xf)(X^m g) && \text{by definition} \\
&= (Xf)(X(g^m \tau_{m;n})) && \text{by lemma 2.17} \\
&= X(f g^m \tau_{m;n}) \\
&= X(g f^n) && \text{since } f \boxtimes g \\
&= (Xg)(Xf)^n \\
&= (Xg)\alpha_{f,X}^n .
\end{aligned}$$

For $F : X \longrightarrow Y$ a morphism of models, we have

$$F\alpha_{f,X} = F(Xf) = (Yf)F^m = \alpha_{f,Y} F^m, \tag{3.41}$$

or in other words, the following diagram commutes:

$$\begin{tikzcd}
X^m \arrow[r, "\alpha_{f,X}"] \arrow[d, "F^m"'] & X \arrow[d, "F"] \\
Y^m \arrow[r, "\alpha_{f,Y}"] & Y.
\end{tikzcd} \tag{3.42}$$

Consequently, the $\alpha_{f,X}$'s assemble into a natural transformation $\alpha_f : \mathrm{id}^m \longrightarrow \mathrm{id}$. Further, the mapping $f \longmapsto \alpha_f$ is injective since $\alpha_{f,K_m} : K_m^m \longrightarrow K_m$ maps id_m to f.

Conversely, let $\beta : \mathrm{id}^m \longrightarrow \mathrm{id}$ be a natural transformation. Let $b \coloneqq \beta_{K_m}(\mathrm{id}_m) : m \longrightarrow 1$. We show that for $X \in \mathsf{K}(\mathcal{S}\mathrm{et})$, the morphism β_X is in fact $Xb : X^m \longrightarrow X$. Let $x \in X^m$, and $\tilde{x} : K_m \longrightarrow X$ be the corresponding morphism under the Yoneda lemma. Then by naturality of β, the following square commutes:

$$\begin{tikzcd}
K_m^m \arrow[r, "\beta_{K_m}"] \arrow[d, "\tilde{x}^m"'] & K_m \arrow[d, "\tilde{x}"] \\
X^m \arrow[r, "\beta_X"] & X.
\end{tikzcd} \tag{3.43}$$

Thus,

$$\beta_X(x) = \beta_X \tilde{x}^m(\mathrm{id}_m) = \tilde{x}\beta_{K_m}(\mathrm{id}_m) = \tilde{x}(a) = (Xa)(x). \tag{3.44}$$

Consequently, $\beta = \alpha_b$, and the map $\alpha : \mathsf{K}(m,1) \longrightarrow [\mathrm{id}^m, \mathrm{id}]$ is a bijection. □

4. Morita theory

Let L be a Lawvere theory. For $k \in \mathbb{N}$, the *matrix theory* [ARV11, definition 15.4] $\mathsf{L}^{[k]}$ is the Lawvere theory where $\mathsf{L}^{[k]}(m,n) = \mathsf{L}(km, kn)$. The structural morphism $\aleph_0^{\mathrm{op}} \longrightarrow \mathsf{L}^{[k]}$ maps the projection $\pi_i : n \longrightarrow 1$ to $\langle \pi_{k(i-1)+1}, \pi_{k(i-1)+2}, \ldots, \pi_{ki} \rangle : kn \longrightarrow k$.

Let now $u : 1 \longrightarrow 1$ be an idempotent operation of L. It is *pseudoinvertible* if there exists $k \in \mathbb{N}$, $a : 1 \longrightarrow k$ and $b : k \longrightarrow 1$ such that

$$bu^k a = \mathrm{id}_1 . \tag{4.1}$$

The *idempotent modification* $u\mathsf{L}u$ of L is the subcategory of L spanned by those morphisms $f : m \longrightarrow n$ such that $fu^m = f = u^n f$. The identity of m in $u\mathsf{L}u$ is u^m, and the i-th projection $m \longrightarrow 1$ is $u\pi_i$.

Lemma 4.2. *The idempotent modification $u\mathsf{L}u$ is generated by morphisms of the form ufu^n, where $f : n \longrightarrow 1$ and $n \in \mathbb{N}$.*

Proof. If $g : m \longrightarrow n$ is in $u\mathsf{L}u$, then

$$g = u^n g u^m = u^n \langle \pi_i g \rangle_{1 \le i \le n} u^m = \langle u(\pi_i g) u^m \rangle_{1 \le i \le n} .$$

□

Two Lawvere theories $\mathsf{K}, \mathsf{L} \in \mathcal{L}\mathrm{aw}$ are *Morita equivalent* [ARV11, definition 15.2], denoted by $\mathsf{K} \sim \mathsf{L}$, if their categories of models $\mathsf{K}(\mathcal{S}\mathrm{et})$ and $\mathsf{L}(\mathcal{S}\mathrm{et})$ are equivalent.

Lemma 4.3. *Let $\mathsf{K}, \mathsf{K}', \mathsf{L}, \mathsf{L}' \in \mathcal{L}\mathrm{aw}$. If $\mathsf{K} \sim \mathsf{K}'$ and $\mathsf{L} \sim \mathsf{L}'$, then $\mathsf{K} \otimes \mathsf{L} \sim \mathsf{K}' \otimes \mathsf{L}'$.*

Proof. Follows from theorem 3.10 and corollary 3.12. □

Proposition 4.4. *Two commutative $\mathsf{K}, \mathsf{L} \in \mathcal{L}\mathrm{aw}$ are Morita equivalent if and only if they are isomorphic.*

Proof. Surely, if K and L are isomorphic, then they are Morita equivalent. Conversely, by proposition 3.40,

$$\mathsf{K}(m,n) = Z(\mathsf{K})(m,n) = [\mathrm{id}^m_{\mathsf{K}(\mathcal{S}\mathrm{et})}, \mathrm{id}^n_{\mathsf{K}(\mathcal{S}\mathrm{et})}] \cong [\mathrm{id}^m_{\mathsf{L}(\mathcal{S}\mathrm{et})}, \mathrm{id}^n_{\mathsf{L}(\mathcal{S}\mathrm{et})}] = \mathsf{L}(m,n). \tag{4.5}$$

□

Theorem 4.6 ([ARV11, theorem 15.7])**.** *Let $\mathsf{K}, \mathsf{L} \in \mathcal{L}\mathrm{aw}$. We have $\mathsf{K} \sim \mathsf{L}$ if and only if $\mathsf{K} \simeq u\mathsf{L}^{[k]}u$, for some $k \ge 1$ and u a pseudoinvertible idempotent of the matrix theory $\mathsf{L}^{[k]}$.*

5. Stability

A Lawvere theory K is said to be *syntactically stable* at rank $k \in \mathbb{N}$ if the canonical map $i_{\mathsf{K}^{\otimes k}} : \mathsf{K}^{\otimes k} \longrightarrow \mathsf{K}^{\otimes k} \otimes \mathsf{K} = \mathsf{K}^{\otimes k+1}$ is an equivalence of categories. Similarly, K is *semantically stable* at rank k if the forgetful functor $i^*_{\mathsf{K}^{\otimes k}} : \mathsf{K}^{\otimes k+1}(\mathcal{S}\mathrm{et}) \longrightarrow \mathsf{K}^{\otimes k}(\mathcal{S}\mathrm{et})$ is an equivalence of categories. Of course, if K is syntactically stable at rank k, then it is semantically stable as well. We say that K is syntactically or semantically stable if it is so at rank 1.

Proposition 5.1. *If K is commutative and semantically stable, then it is syntactically stable.*

Proof. Follows from lemma 3.39 and proposition 4.4. □

Proposition 5.2. *If K is syntactically stable, then it is commutative.*

Proof. Denote by $i_1, i_2 : \mathsf{K} \longrightarrow \mathsf{K} \otimes \mathsf{K}$ the canonical morphisms into the left and right component, respectively, and let $\sigma : \mathsf{K} \otimes \mathsf{K} \longrightarrow \mathsf{K} \otimes \mathsf{K}$ be the symmetry involution, i.e. such that $\sigma i_1 = i_2$ and conversely. Then by assumption, i_1 is an isomorphisms, and thus, so is i_2. By definition, for $f, g \in \mathsf{K}/1$, we have $i_1(f) \boxtimes i_2(g)$, and applying i_1^{-1} gives $f \boxtimes i_1^{-1} i_2(g)$. Note that $i_1^{-1} i_2 = i_1^{-1} \sigma i_1 = i_2^{-1} i_1$ is an involution. In particular,

$$f \boxtimes i_1^{-1} i_2 (i_1^{-1} i_2(g)) = g, \tag{5.3}$$

for all $f, g \in \mathsf{K}/1$. $\square$

Unfortunately, the converse of proposition 5.2 does not hold, see e.g. Mag_1.

6. Conclusion

TODO: Write the conclusion.

References

[AR94] Jiří Adámek and Jiří Rosický. *Locally Presentable and Accessible Categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[ARV11] Jiří Adámek, Jiří Rosický, and E. M. Vitale. *Algebraic Theories: A Categorical Introduction to General Algebra*. Number 184 in Cambridge Tracts in Mathematics. Cambridge Univ. Press, Cambridge, 2011.

[Bor94a] Francis Borceux. *Handbook of Categorical Algebra. 1*, volume 50 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, Cambridge, 1994.

[Bor94b] Francis Borceux. *Handbook of Categorical Algebra. 2*, volume 51 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, Cambridge, 1994.

[Bor94c] Francis Borceux. *Handbook of Categorical Algebra. 3*, volume 52 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, Cambridge, 1994.

[BV79] J.M. Boardman and R.M. Vogt. Tensor products of theories, application to infinite loop spaces. *Journal of Pure and Applied Algebra*, 14(2):117–129, March 1979.

[BV94] Francis Borceux and Enrico M. Vitale. On the notion of bimodel for functorial semantics. *Applied Categorical Structures*, 2(3):283–295, 1994.

[Che20] Eugenia Cheng. Distributive laws for Lawvere theories. *Compositionality*, 2:1, May 2020.

[EH62] B. Eckmann and P. J. Hilton. Group-like structures in general categories I multiplications and comultiplications. *Mathematische Annalen*, 145(3):227–255, June 1962.

[Fre66] P. Freyd. Algebra valued functors in general and tensor products in particular. *Colloquium Mathematicum*, 14(1):89–106, 1966.

[Fuj19] Soichiro Fujii. A unified framework for notions of algebraic theory. *Theory and Applications of Categories*, 34:Paper No. 40, 1246–1316, 2019.

[HP07] Martin Hyland and John Power. The category theoretic understanding of universal algebra: Lawvere theories and monads. In *Computation, Meaning, and Logic: Articles Dedicated to Gordon Plotkin*, volume 172 of *Electron. Notes Theor. Comput. Sci.*, pages 437–458. Elsevier Sci. B. V., Amsterdam, 2007.

[Law04] F. William Lawvere. Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theoriesc. *Reprints in Theory and Applications of Categories*, (5):1–121, 2004.

[LP09] Stephen Lack and John Power. Gabriel-Ulmer duality and Lawvere theories enriched over a general base. *Journal of Functional Programming*, 19(3-4):265–286, 2009.